

## CHAPTER 2

# MATHEMATICAL MODELS AND NUMERICAL METHODS

### SECTION 2.1

### POPULATION MODELS

Section 2.1 introduces the first of the two major classes of mathematical models studied in the textbook, and is a prerequisite to the discussion of equilibrium solutions and stability in Section 2.2. In Problems 1-8 we find the desired particular solution and sketch some typical solution curves, with the desired particular solution highlighted.

1. Separating variables gives  $\int \frac{1}{x(1-x)} dx = \int dt$ . By the method of partial fractions

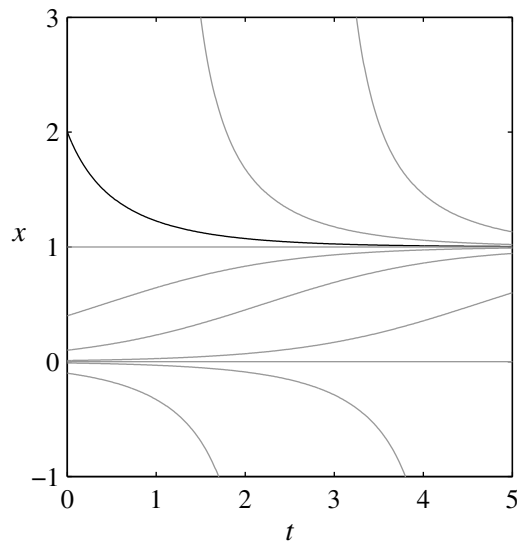
$$\int \frac{1}{x(1-x)} dx = \int \frac{1}{x} - \frac{1}{x-1} dx = \ln|x| - \ln|x-1|,$$

and so the general solution of the differential equation is  $\ln|x| - \ln|x-1| = t + C$ , or

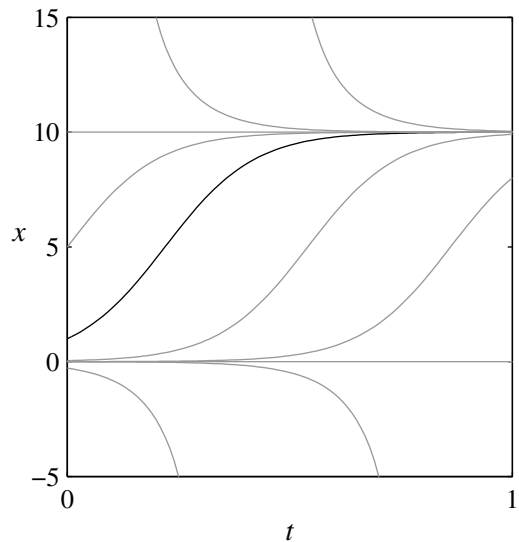
$\frac{x}{x-1} = Ce^t$ . The initial condition  $x(0) = 2$  implies that  $C = 2$ , leading to the particular

solution  $\frac{x}{x-1} = 2e^t$ , or  $x = 2(x-1)e^t$ , or finally  $x(t) = \frac{2e^t}{2e^t - 1} = \frac{2}{2 - e^{-t}}$ .

**Problem 1**



**Problem 2**



2. Separating variables gives  $\int \frac{1}{x(10-x)} dx = \int dt$ . By the method of partial fractions

$$\int \frac{1}{x(10-x)} dx = \frac{1}{10} \int \frac{1}{x} - \frac{1}{x-10} dx = \frac{1}{10} (\ln|x| - \ln|x-10|),$$

and so the general solution of the differential equation is  $\ln|x| - \ln|x-10| = 10t + C$ , or

$\frac{x}{x-10} = Ce^{10t}$ . The initial condition  $x(0) = 1$  implies that  $C = -\frac{1}{9}$ , leading to the

particular solution  $\frac{x}{10-x} = \frac{1}{9}e^{10t}$ , or  $9x = e^{10t}(10-x)$ , or finally

$$x(t) = \frac{10e^{10t}}{e^{10t} + 9} = \frac{10}{1 + 9e^{-10t}}.$$

3. Separating variables gives  $\int \frac{1}{(1+x)(1-x)} dx = \int dt$ . By the method of partial fractions

$$\int \frac{1}{(x+1)(x-1)} dx = -\frac{1}{2} \int \frac{1}{x-1} - \frac{1}{x+1} dx = -\frac{1}{2} (\ln|x-1| - \ln|x+1|),$$

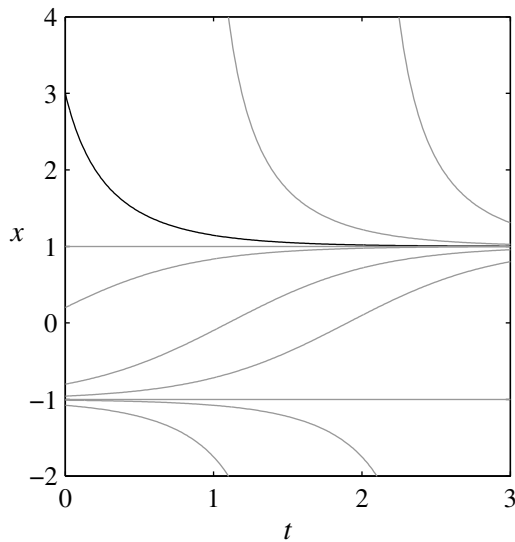
and so the general solution of the differential equation is  $\ln|x-1| - \ln|x+1| = -2t + C$ , or

$\frac{x-1}{x+1} = Ce^{-2t}$ . The initial condition  $x(0) = 3$  implies that that  $C = \frac{1}{2}$ , leading to the

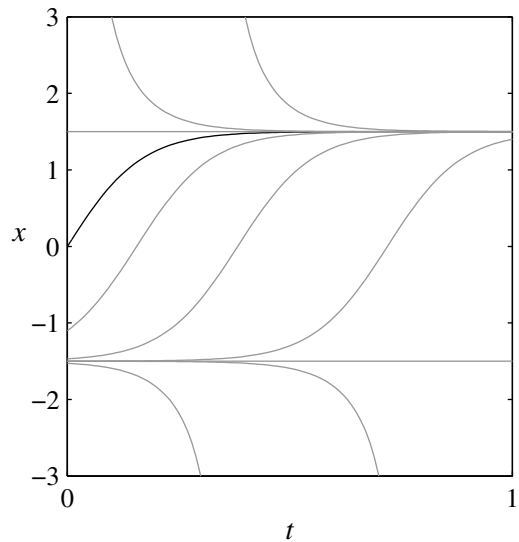
particular solution  $\frac{x-1}{x+1} = \frac{1}{2}e^{-2t}$ , or  $2(x-1) = (x+1)e^{-2t}$ , or finally

$$x(t) = \frac{2 + e^{-2t}}{2 - e^{-2t}} = \frac{2e^{2t} + 1}{2e^{2t} - 1}.$$

**Problem 3**



**Problem 4**



4. Separating variables gives  $\int \frac{1}{(3+2x)(3-2x)} dx = \int dt$ . By the method of partial fractions,

$$\int \frac{1}{(3+2x)(3-2x)} dx = \frac{1}{6} \int \frac{1}{3+2x} - \frac{1}{3-2x} dx = \frac{1}{12} (\ln|3+2x| - \ln|3-2x|),$$

and so the general solution of the differential equation is  $\ln|3+2x| - \ln|3-2x| = 12t + C$ ,

or  $\frac{3+2x}{3-2x} = Ce^{12t}$ . The initial condition  $x(0) = 0$  implies that  $C = 1$ , leading to the

particular solution  $\frac{3+2x}{3-2x} = e^{12t}$ , or  $3+2x = (3-2x)e^{12t}$ , or finally

$$x(t) = \frac{3e^{12t} - 3}{2e^{12t} + 2} = \frac{3(e^{12t} - 1)}{2(e^{12t} + 1)}.$$

5. Separating variables gives  $\int \frac{1}{x(x-5)} dx = \int -3 dt$ . By the method of partial fractions,

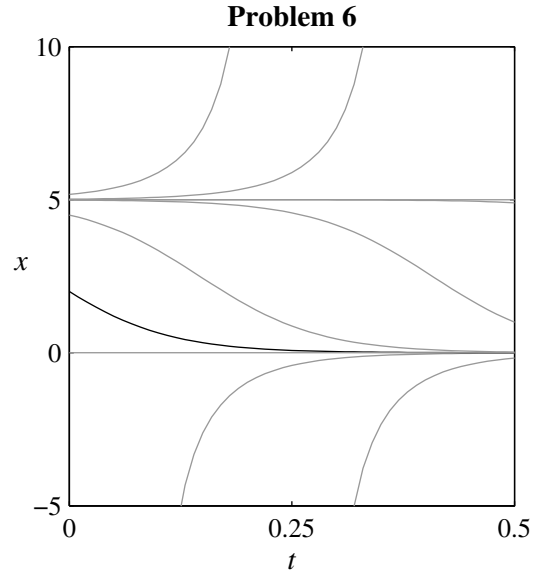
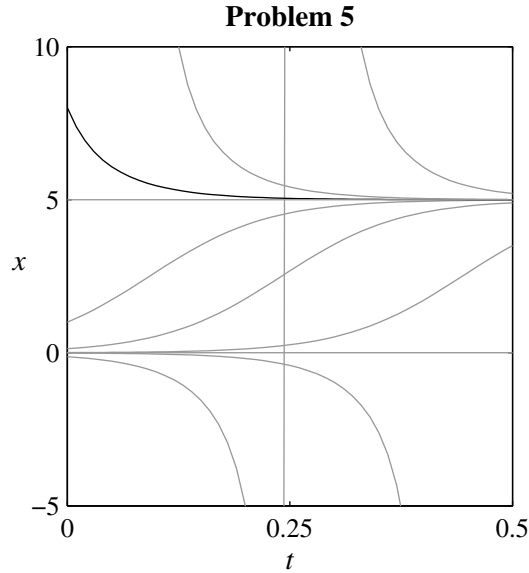
$$\int \frac{1}{x(x-5)} dx = -\frac{1}{5} \int \frac{1}{x} - \frac{1}{x-5} dx = -\frac{1}{5} (\ln|x| - \ln|x-5|),$$

and so the general solution of the differential equation is  $-\frac{1}{5} (\ln|x| - \ln|x-5|) = -3t + C$ ,

or  $\frac{x}{x-5} = Ce^{15t}$ . The initial condition  $x(0) = 8$  implies that  $C = \frac{8}{3}$ , leading to the

particular solution  $\frac{x}{x-5} = \frac{8}{3} e^{15t}$ , or  $3x = 8(x-5)e^{15t}$ , or finally

$$x(t) = \frac{-40e^{15t}}{3-8e^{15t}} = \frac{40}{8-3e^{-15t}}.$$



6. Separating variables gives  $\int \frac{1}{x(x-5)} dx = \int 3 dt$ . Using the partial fraction expansion

found in Problem 5, we find the general solution  $-\frac{1}{5}(\ln|x| - \ln|x-5|) = 3t + C$ , or

$\frac{x}{x-5} = Ce^{-15t}$ . The initial condition  $x(0) = 2$  implies that  $C = \frac{2}{3}$ , leading to the

particular solution  $\frac{x}{x-5} = \frac{2}{3}e^{-15t}$ , or  $3x = 2(x-5)e^{-15t}$ , or finally

$$x(t) = \frac{10e^{-15t}}{3 + 2e^{-15t}} = \frac{10}{2 + 3e^{15t}}.$$

7. Separating variables gives  $\int \frac{1}{x(x-7)} dx = \int -4 dt$ . By the method of partial fractions,

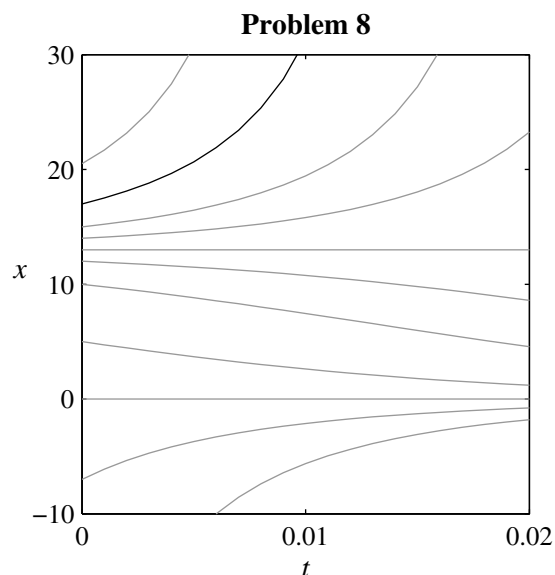
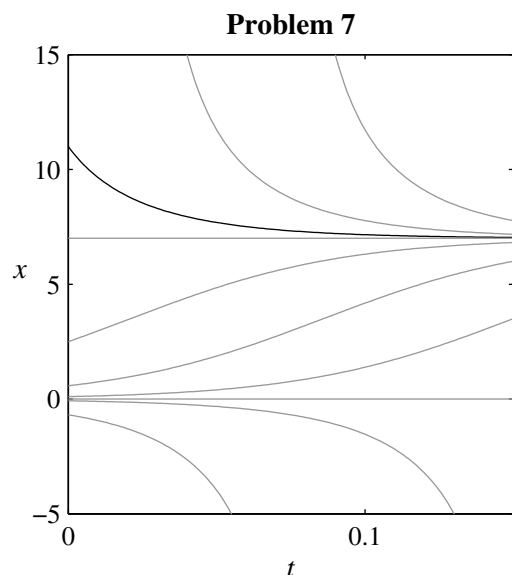
$$\int \frac{1}{x(x-7)} dx = -\frac{1}{7} \int \frac{1}{x} - \frac{1}{x-7} dx = -\frac{1}{7}(\ln|x| - \ln|x-7|),$$

and so the general solution of the differential equation is  $\ln|x| - \ln|x-7| = 28t + C$ , or

$\frac{x}{x-7} = Ce^{28t}$ . The initial condition  $x(0) = 11$  implies that  $C = \frac{11}{4}$ , leading to the

particular solution  $\frac{x}{x-7} = \frac{11}{4}e^{28t}$ , or  $4x = 11(x-7)e^{28t}$ , or finally

$$x(t) = \frac{-77e^{28t}}{4 - 11e^{28t}} = \frac{77}{11 - 4e^{-28t}}.$$



8. Separating variables gives  $\int \frac{1}{x(x-13)} dx = \int 7 dt$ . By the method of partial fractions,

$$\int \frac{1}{x(x-13)} dx = -\frac{1}{13} \int \frac{1}{x} - \frac{1}{x-13} dx = -\frac{1}{13} (\ln|x| - \ln|x-13|),$$

and so the general solution of the differential equation is  $\ln|x| - \ln|x-13| = -91t + C$ , or

$\frac{x}{x-13} = Ce^{-91t}$ . The initial condition  $x(0) = 17$  implies that  $C = \frac{17}{4}$ , leading to the

particular solution  $\frac{x}{x-13} = \frac{17}{4}e^{-91t}$ , or  $4x = 17(x-13)e^{-91t}$ , or finally

$$x(t) = \frac{-221e^{-91t}}{4-17e^{-91t}} = \frac{221}{17-4e^{91t}}.$$

9. Substitution of  $P(0) = 100$  and  $P'(0) = 20$  into  $P' = k\sqrt{P}$  yields  $k = 2$ , so the differential equation is  $P' = 2\sqrt{P}$ . Separation of variables gives  $\int \frac{1}{2\sqrt{P}} dP = \int dt$ , which upon integrating is  $\sqrt{P} = t + C$ . Then  $P(0) = 100$  implies  $C = 10$ , so that  $P(t) = (t+10)^2$ . Hence the number of rabbits after one year is  $P(12) = 484$ .

10. Given that  $P' = -\delta P = -\frac{k}{\sqrt{P}}P = -k\sqrt{P}$ , separation of variables and integration as in Problem 9 yields  $2\sqrt{P} = -kt + C$ . The initial condition  $P(0) = 900$  gives  $C = 60$ , and then the condition  $P(6) = 441$  implies that  $k = 3$ . Therefore  $2\sqrt{P} = -3t + 60$ , so that  $P = 0$  after  $t = 20$  weeks.

11. (a) Substituting our assumptions that  $\beta = \frac{k_1}{\sqrt{P}}$  and  $\delta = \frac{k_2}{\sqrt{P}}$  into the general population equation gives  $\frac{dP}{dt} = \frac{(k_1 - k_2)}{\sqrt{P}} \cdot P = k\sqrt{P}$ . Separation of variables leads to  $\int \frac{1}{\sqrt{P}} dP = \int k dt$ , which upon integrating is  $2\sqrt{P} = kt + C$ , or  $P = \left(\frac{kt}{2} + C\right)^2$ . The initial condition  $P(0) = P_0$  then gives  $C = \sqrt{P_0}$ .
- (b) Our assumption implies that  $C = \sqrt{P_0} = 10$ , so that  $P = \left(\frac{kt}{2} + 10\right)^2$ . Measuring  $t$  in months, we conclude from  $P(6) = 169$  that  $k = 1$ , so that  $P = \left(\frac{t}{2} + 10\right)^2$ . Hence there are  $P(12) = 256$  fish after 12 months.
12. Separating variables in our assumption that  $P' = kP^2$  gives  $\int \frac{1}{P^2} dP = \int k dt$ , which upon integrating leads to  $-\frac{1}{P} = kt + C$ , or  $P = \frac{1}{C - kt}$ . Now  $P(0) = 12$  implies that  $C = \frac{1}{12}$ , where  $t$  is measured in years, so that  $P = \frac{12}{1 - 12kt}$ . Then  $P(10) = 24$  gives  $k = \frac{1}{240}$ , so that  $P(t) = \frac{240}{20 - t}$ . Thus  $P(t) = 48$  when  $t = 15$ , that is, in the year 2003. Finally, under these assumptions the alligator population will grow without bound as  $t$  approaches 20 years, that is, the year 2008.
13. (a) Substituting our assumptions that  $\beta = k_1P$  and  $\delta = k_2P$  into the general population equation gives  $\frac{dP}{dt} = [(k_1 - k_2)P]P = kP^2$ , where  $k = k_1 - k_2 > 0$  by our assumption that  $\beta > \delta$ . Solving as in Problem 12 leads to  $P = \frac{1}{C - kt}$ . The initial condition  $P(0) = P_0$  implies that  $C = \frac{1}{P_0}$ , so that  $P(t) = \frac{P_0}{1 - kP_0t}$ . As  $t \rightarrow \frac{1}{kP_0}$  we find that  $P(t) \rightarrow \infty$ .
- (b) Our assumption that  $P_0 = 6$  gives  $P(t) = \frac{6}{1 - 6kt}$ . Then, with  $t$  measured in months, we conclude from  $P(10) = 9$  that  $k = \frac{1}{180}$ , so that  $P(t) = \frac{6}{1 - (t/30)} = \frac{180}{30 - t}$ . From this we can see that doomsday occurs after 30 months.

14. Now  $\frac{dP}{dt} = kP^2$  with  $k < 0$ , and solving once again leads to  $P(t) = \frac{P_0}{1 - kP_0t}$ . As  $t \rightarrow \infty$  the rabbit population  $P$  approaches zero, because  $k$  is negative.
15. Writing  $\frac{dP}{dt} = bP\left(\frac{a}{b} - P\right)$  shows that the limiting population  $M$  is  $\frac{a}{b}$ . Then the facts that  $B_0 = aP_0$  and  $D_0 = aP_0^2$  give  $\frac{B_0P_0}{D_0} = \frac{(aP_0)P_0}{bP_0^2} = \frac{a}{b} = M$ . With Problems 16 and 17 in mind, we note also that  $a = \frac{B_0}{P_0}$  and  $b = \frac{D_0}{P_0^2} = k$ .
16. The relations in Problem 15 give  $k = \frac{D_0}{P_0^2} = \frac{6}{120^2} = \frac{1}{2400}$  and a limiting population of  $M = \frac{B_0P_0}{D_0} = \frac{8 \cdot 120}{6} = 160$  rabbits. By Equation (7), proved below in Problem 32, the solution is  $P(t) = \frac{160 \cdot 120}{120 + (160 - 120)e^{-t/15}} = \frac{19200}{120 + 40e^{-t/15}}$ . Setting  $P(t) = 0.95M = 152$  rabbits yields  $t \approx 27.69$  months.
17. The relations in Problem 15 give  $k = \frac{D_0}{P_0^2} = \frac{12}{240^2} = \frac{1}{2400}$  and a limiting population of  $M = \frac{B_0P_0}{D_0} = \frac{9 \cdot 240}{12} = 180$  rabbits. The solution is then 
$$P(t) = \frac{180 \cdot 240}{240 + (180 - 240)e^{-t/15}} = \frac{43200}{120 - 60e^{-t/15}},$$
 again by Equation (7). Setting  $P(t) = 1.05M = 189$  rabbits yields  $t \approx 44.22$  months.
18. Writing  $\frac{dP}{dt} = aP\left(P - \frac{b}{a}\right)$  shows that the limiting population  $M$  is  $\frac{b}{a}$ . Then the facts that  $B_0 = aP_0^2$  and  $D_0 = aP_0$  give  $\frac{D_0P_0}{B_0} = \frac{(bP_0)P_0}{aP_0^2} = \frac{b}{a} = M$ . With Problems 19 and 20 in mind, we note also that  $b = \frac{D_0}{P_0}$  and  $a = \frac{B_0}{P_0^2} = k$ .
19. The relations in Problem 18 give  $k = \frac{B_0}{P_0^2} = \frac{10}{100^2} = \frac{1}{1000}$  and  $M = \frac{D_0P_0}{B_0} = \frac{9 \cdot 100}{10} = 90$ . Problem 33 below then gives the solution

$$P(t) = \frac{90 \cdot 100}{100 + (90 - 100)e^{9t/100}} = \frac{9000}{100 - 10e^{9t/100}}.$$

Setting  $P(t) = 10M = 900$  rabbits yields  $t \approx 24.41$  months.

20. The relations in Problem 18 give  $k = \frac{B_0}{P_0^2} = \frac{11}{110^2} = \frac{1}{1100}$  and  $M = \frac{D_0 P_0}{B_0} = \frac{12 \cdot 110}{11} = 120$ .

Problem 33 below then gives the solution

$$P(t) = \frac{120 \cdot 110}{110 + (120 - 110)e^{120t/1100}} = \frac{13200}{110 + 10e^{6t/55}}.$$

Setting  $P(t) = 0.1M = 12$  rabbits yields  $t \approx 42.12$  months.

21. Separating variables in our assumption that  $\frac{dP}{dt} = kP(200 - P)$  gives

$$\int \frac{1}{P(200 - P)} dP = \int k dt. \text{ By the method of partial fractions}$$

$$\int \frac{1}{P(200 - P)} dP = \frac{1}{200} \int \frac{1}{P} + \frac{1}{200 - P} dP = \frac{1}{200} (\ln P - \ln |200 - P|),$$

and so the general solution of the differential equation is  $\ln P - \ln |200 - P| = 200kt$ , or

$$\ln \frac{P}{|200 - P|} = 200kt + C, \text{ or } \frac{P}{200 - P} = Ce^{200kt}. \text{ The initial condition } P(0) = 100 \text{ (taking}$$

$t = 0$  in 1960) implies that  $C = 1$ . Further,  $P'(0) = 1$ , when substituted into the original differential equation along with  $P(0) = 100$ , implies that  $1 = k \cdot 100(200 - 100)$ , or

$$k = \frac{1}{10000}. \text{ Substituting these values into the general solution gives } \frac{P}{200 - P} = e^{t/50}, \text{ or}$$

$$P = e^{t/50} (200 - P), \text{ or } P(t) = \frac{200}{1 + e^{-t/50}}. \text{ Finally, in the year 2020 the country's}$$

population will be  $P(60) = \frac{200}{1 + e^{-6/5}} \approx 153.7$  million.

22. We work in thousands of persons, and so take  $M = 100$  for the total fixed population. Substituting this together with  $P_0 = 50$  and  $P'(0) = 1$  into the logistic equation gives  $1 = k \cdot 50(100 - 50)$ , or  $k = 0.0004$ . If  $t$  denotes the number of days until 80 thousand people have heard the rumor, then Eq. (7) in the text gives

$$80 = \frac{50 \cdot 100}{50 + (100 - 50)e^{-0.04t}},$$

and we solve this equation to find  $t \approx 34.66$ . Thus the rumor will have spread to 80% of the population in a little less than 35 days.



23. (a) The given differential equation implies that

$$x' = 0.8x - 0.004x^2 = 0.004x(200 - x),$$

which is positive for  $0 < x < 200$  and negative for  $x > 200$ ; thus the maximum amount that will dissolve is  $M = 200$  g.

(b) Since the given equation conforms to Equation (6) in the text, the solution is given there by Equation (7), with  $M = 200$ ,  $P_0 = 50$ , and  $k = 0.004$ :

$$x(t) = \frac{10000}{50 + 150e^{-0.8t}}.$$

Substituting  $x = 100$ , we solve for  $t = 1.25 \ln 3 \approx 1.37$  sec.

24. Our assumptions imply that  $N'(t) = kN(15 - N)$ , where we measure  $N$  in thousands of people. Substituting  $N(0) = 5$  and  $N'(0) = 0.5$  gives  $k = 0.01$ . With  $N$  in place of  $P$ , this is the logistic initial value problem in Equation (6) of the text (using  $M = 15$ ), so its solution is given by Equation (7):

$$N(t) = \frac{15 \cdot 5}{5 + (15 - 5)e^{-0.0115t}} = \frac{15}{1 + 2e^{-0.15t}}.$$

For another 5000 people to develop the syndrome means that a total of 10,000 people are afflicted, that is,  $N = 10$ . Upon substituting, we solve for  $t = \frac{\ln 4}{0.15} \approx 9.24$  days.

25. (a) Following the suggestions (and thus taking  $t = 0$  in 1925), we estimate the rate of population growth in 1925 to be

$$P'(0) = \frac{P(1) - P(-1)}{2} = \frac{25.38 - 24.63}{2} = 0.375$$

million people annually. The corresponding estimate for the year 1975, corresponding to  $t = 50$ , is

$$P'(50) = \frac{P(51) - P(49)}{2} = \frac{48.04 - 47.04}{2} = 0.5$$

million people annually. Substituting these values, together with  $P(0) = 25$  and  $P(50) = 47.54$ , into the logistic equation (3) leads to the system of equations

$$\begin{aligned} 0.375 &= 25k(M - 25) \\ 0.5 &= 47.54k(M - 47.54) \end{aligned}$$

As in Example 3 in the text, we solve these equations to find  $M = 100$  and  $k = 0.0002$ . Then Equation (7) gives the population function

$$P(t) = \frac{100 \cdot 25}{25 + (100 - 25)e^{-0.0002 \cdot 100t}} = \frac{2500}{25 + 75e^{-0.02t}}.$$

- (b) We find that  $P = 75$  when  $t = 50 \ln 9 \approx 110$ , that is, in 2035 A.D.

26. Our assumptions lead to the differential equation  $\frac{dP}{dt} = 0.001P^2 - \delta P$  for the rodent population  $P(t)$ . Substituting  $P(0) = 100$  and  $P'(0) = 8$  gives  $\delta = 0.02$ , and so

$$\frac{dP}{dt} = 0.001P^2 - 0.02P = 0.001P(P - 20).$$

Separating variables gives  $\int \frac{1}{P(P - 20)} dP = \int 0.001 dt$ . By the method of partial fractions

$$\int \frac{1}{P(P - 20)} dP = -\frac{1}{20} \int \frac{1}{P} - \frac{1}{P - 20} dP = \frac{1}{20} (\ln|P - 20| - \ln P),$$

and so the general solution of the differential equation is

$$\frac{1}{20} \ln|P - 20| - \ln P = 0.001t + C,$$

or  $\ln \frac{|P - 20|}{P} = \frac{t}{50} + C$ , or  $\frac{P - 20}{P} = Ce^{t/50}$ . The initial condition  $P(0) = 100$  implies that

$C = \frac{4}{5}$ , so that  $\frac{P - 20}{P} = \frac{4}{5}e^{t/50}$ , and then solving for  $P$  gives  $P(t) = \frac{100}{5 - 4e^{t/50}}$ . Finally,

setting  $P(t) = 200$  leads to  $t = 50 \ln \frac{9}{8} \approx 5.89$  months.

27. Our assumptions lead to the differential equation  $\frac{dP}{dt} = kP^2 - 0.01P$  for the animal population  $P(t)$ . Substituting  $P(0) = 200$  and  $P'(0) = 2$ , we find that  $k = 0.0001$ , so that

$$\frac{dP}{dt} = 0.0001P^2 - 0.01P = 0.0001P(P - 100).$$

Separating variables gives  $\int \frac{1}{P(P - 100)} dP = \int 0.0001 dt$ . By the method of partial fractions

$$\int \frac{1}{P(P - 100)} dP = -\frac{1}{100} \int \frac{1}{P} - \frac{1}{P - 100} dP = \frac{1}{100} (\ln|P - 100| - \ln P),$$

and so the general solution of the differential equation is

$$\frac{1}{100} \ln|P - 100| - \ln P = 0.0001t + C,$$

or  $\ln \frac{|P - 100|}{P} = \frac{t}{100} + C$ , or  $\frac{P - 100}{P} = Ce^{t/100}$ . The initial condition  $P(0) = 200$  gives

$C = \frac{1}{2}$ , and so  $\frac{P - 100}{P} = \frac{1}{2}e^{t/100}$ , leading to the general solution  $P(t) = \frac{200}{2 - e^{t/100}}$ .

(a) Setting  $P = 1000$  gives  $t = 100 \ln \frac{9}{5} \approx 58.78$  months.

(b) Doomsday occurs as the denominator  $2 - e^{t/100}$  approaches zero, that is, as  $t$  approaches  $100 \ln 2 \approx 69.31$  months, since the population  $P$  becomes infinite then.

28. Our alligator population satisfies the equation

$$\frac{dx}{dt} = 0.0001x^2 - 0.01x = 0.0001x(x - 100).$$

With  $x$  in place of  $P$ , this is the same differential equation as in Problem 27, and so our general solution is  $\frac{x-100}{x} = Ce^{t/100}$ , as found there.

(a) The assumption  $x(0) = 25$  gives  $C = -3$ , leading to  $x(t) = \frac{100}{1 + 3e^{t/100}}$ , which approaches zero as  $t$  increases without bound. Thus the alligator population faces extinction in this event.

(b) If instead  $x(0) = 150$ , however, then we find that  $C = \frac{1}{3}$ , leading to  $x(t) = \frac{300}{3 - e^{t/100}}$ . Now find that  $x(t)$  grows without bound as  $t$  approaches  $100 \ln 3 \approx 110$  months, that is, doomsday occurs at this time.

29. Here we have the logistic equation

$$\frac{dP}{dt} = 0.03135P - 0.0001489P^2 = 0.0001489P(210.544 - P),$$

where  $k = 0.0001489$  and  $P = 210.544$ . With  $P_0 = 3.9$  as well, Eq. (7) in the text gives

$$P(t) = \frac{(210.544)(3.9)}{3.9 + (210.544 - 3.9)e^{-(0.0001489)(210.544)t}} = \frac{821.122}{3.9 + 206.644e^{-0.03135t}}.$$

(a) This solution gives  $P(140) \approx 127.008$ , fairly close to the actual 1930 U.S. census figure of 123.2 million.

(b) As  $t$  grows without bound,  $P(t)$  approaches  $\frac{821.122}{3.9} = 210.544$  million.

(c) Since the actual U.S. population in 2000 was about 281 million—already exceeding the maximum population predicted by the logistic equation—we see that that this model did *not* continue to hold throughout the 20th century.

30. Separating variables in the differential equation gives  $\int \frac{1}{P} dP = \int \beta_0 e^{-\alpha t} dt$ , with general solution  $\ln P = -\frac{\beta_0}{\alpha} e^{-\alpha t} + C$ . The initial condition  $P(0) = P_0$  gives  $C = \ln P_0 + \frac{\beta_0}{\alpha}$ ,

leading to the desired solution  $P(t) = P_0 \exp\left[\frac{\beta_0}{\alpha}(1 - e^{-\alpha t})\right]$ , with limiting population  $P_0 \exp\left(\frac{\beta_0}{\alpha}\right)$  as  $t$  grows without bound.

- 31.** Substituting  $P(0) = 10^6$  and  $P'(0) = 3 \times 10^5$  into the differential equation  $P'(t) = \beta_0 e^{-\alpha t} P$  yields  $\beta_0 = 0.3$ . Hence the solution given in Problem 30 is  $P(t) = P_0 \exp\left[\frac{0.3}{\alpha}(1 - e^{-\alpha t})\right]$ . The fact that  $P(6) = 2P_0$  now yields the equation

$$0.3(1 - e^{-6\alpha}) - \alpha \ln 2 = 0,$$

which we seek to solve for the constant  $\alpha$ . We let  $f(\alpha)$  denote the left-hand side  $0.3(1 - e^{-6\alpha}) - \alpha \ln 2$  of this equation and apply Newton's iterative formula

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$$

with initial guess  $\alpha_0 = 1$  (suggested by a plot of  $f(\alpha)$ ), leading quickly to  $\alpha \approx 0.3915$ . Therefore the limiting cell population as  $t$  grows without bound is

$$P_0 \exp\left(\frac{\beta_0}{\alpha}\right) = 10^6 \exp\left(\frac{0.3}{0.3915}\right) \approx 2.15 \times 10^6.$$

Thus the tumor does not grow much further after 6 months.

- 32.** Separating variables in the logistic equation gives  $\int \frac{1}{P(M-P)} dP = \int k dt$ . By the method of partial fractions

$$\int \frac{1}{P(M-P)} dP = \frac{1}{M} \int \frac{1}{P} + \frac{1}{M-P} dP = \frac{1}{M} (\ln P - \ln |M-P|),$$

and so the general solution is

$$\frac{1}{M} (\ln P - \ln |M-P|) = kt + C,$$

or  $\ln \frac{P}{|M-P|} = kMt + C$ , or  $\frac{P}{|M-P|} = Ce^{kMt}$ . The initial condition  $P(0) = P_0$  gives

$C = \frac{P_0}{|M-P_0|}$ . If the initial population  $P_0$  is less than the limiting population  $M$ , then

$C = \frac{P_0}{M-P_0}$ . Moreover, in this event  $P < M$  for all  $t$ , since the logistic equation itself

shows that the only solution  $P(t)$  that takes on the value  $M$  is the constant solution

$P(t) \equiv M$ . It follows that if  $P_0 < M$ , then the solution of the logistic initial value problem is  $\frac{P}{M-P} = \frac{P_0}{M-P_0} e^{kMt}$ . Similarly, if  $P_0 > M$ , then  $P > M$  for all  $t$ , leading to the solution  $\frac{P}{P-M} = \frac{P_0}{P_0-M} e^{kMt}$ , which is equivalent to the solution for the preceding case  $P_0 < M$ . Solving either of these for  $P$  yields

$$P(M - P_0) = P_0(M - P)e^{kMt},$$

or

$$(P_0 e^{kMt} + M - P_0)P = MP_0 e^{kMt},$$

or finally

$$P(t) = \frac{MP_0 e^{kMt}}{P_0 e^{kMt} + M - P_0} = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}.$$

**33. (a)** Separating variables in the extinction-explosion equation gives

$$\int \frac{1}{P(P-M)} dP = \int k dt. \text{ By the method of partial fractions}$$

$$\int \frac{1}{P(P-M)} dP = \frac{1}{M} \int \frac{1}{P-M} - \frac{1}{P} dP = \frac{1}{M} (\ln|P-M| - \ln P),$$

and so the general solution is

$$\frac{1}{M} (\ln|P-M| - \ln P) = kt + C,$$

or  $\ln \frac{|P-M|}{P} = kMt + C$ , or  $\frac{|P-M|}{P} = Ce^{kMt}$ . The initial condition  $P(0) = P_0$  gives

$C = \frac{|P_0 - M|}{P_0}$ . If the initial population  $P_0$  is less than the threshold population  $M$ , then

$C = \frac{M - P_0}{P_0}$ . Moreover, as in Problem 32, in this case  $P < M$  for all  $t$ . Thus for  $P_0 < M$

the solution of the extinction-explosion initial value problem is  $\frac{M-P}{P} = \frac{M-P_0}{P_0} e^{kMt}$ .

Similarly, if  $P_0 > M$ , then  $P > M$  for all  $t$ , and so the solution is  $\frac{P-M}{P} = \frac{P_0-M}{P_0} e^{kMt}$ .

Solving either of these equivalent expressions for  $P$  yields

$$P_0(P - M) = P(P_0 - M)e^{kMt},$$

or

$$[P_0 + (M - P_0)e^{kMt}]P = MP_0,$$

or finally

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{kMt}}.$$

(b) If  $P_0 < M$ , then the coefficient  $M - P_0$  is positive and the denominator increases without bound, so  $P(t) \rightarrow 0$  as  $t \rightarrow \infty$ . But if  $P_0 > M$ , then the denominator  $P_0 - (P_0 - M)e^{kMt}$  approaches zero—so  $P(t) \rightarrow +\infty$ —as  $t$  approaches the positive value  $\frac{1}{kM} \ln \frac{P_0}{P_0 - M}$  from the left. Thus the population either becomes extinct or explodes.

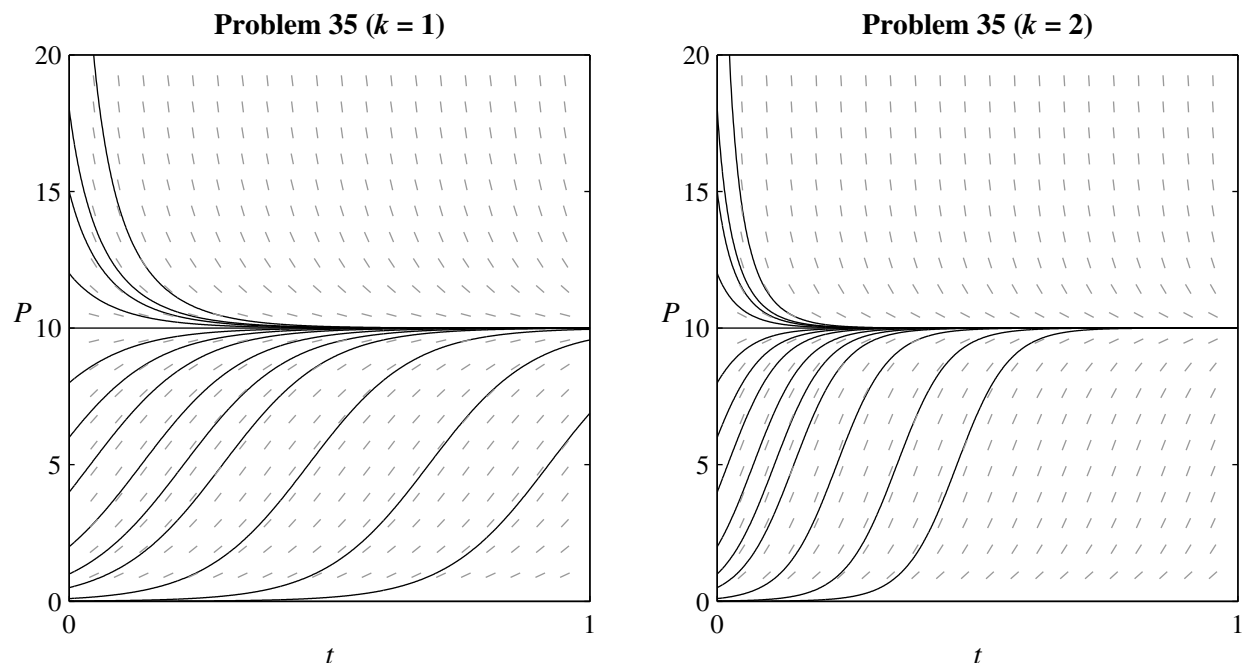
34. Differentiation of both sides of the logistic equation  $P' = kP(M - P)$  yields

$$\begin{aligned} P'' &= \frac{dP'}{dP} \cdot \frac{dP}{dt} \\ &= [k \cdot (M - P) + kP \cdot (-1)] \cdot kP(M - P) \\ &= k(M - 2P) \cdot kP(M - P) \\ &= 2k^2P \left( M - \frac{1}{2}P \right) (M - P), \end{aligned}$$

as desired. The conclusions that  $P'' > 0$  if  $0 < P < \frac{1}{2}M$ , that  $P'' = 0$  if  $P = \frac{1}{2}M$ , and that  $P'' < 0$  if  $\frac{1}{2}M < P < M$  are then immediate. Thus it follows that each of the curves for which  $P_0 < M$  has an inflection point where it crosses the horizontal line  $P = \frac{1}{2}M$ .

35. Any way you look at it, you should conclude that the larger the parameter  $k > 0$ , the faster the logistic population  $P(t)$  approaches its limiting population  $M$ :

To examine the question *geometrically*, we will assume that  $M = 10$  and that  $k_1 = 1$  and  $k_2 = 2$ , leading to the logistic equations  $\frac{dP}{dt} = P(10 - P)$  and  $\frac{dP}{dt} = 2P(10 - P)$ . We draw slope fields and solution curves for each of these equations, using the same initial values  $P(0)$  in both cases:



These diagrams suggest that the larger the value of  $k$ , the more rapidly the population  $P(t)$  approaches the limiting population  $M$ .

To look at things *analytically*, we examine the distance between the solution (7) in the text of the logistic initial value problem and the limiting population  $M$ :

$$\left| M - \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}} \right| = \left| \frac{M[P_0 + (M - P_0)e^{-kMt}] - MP_0}{P_0 + (M - P_0)e^{-kMt}} \right| = \frac{M|M - P_0|}{P_0(e^{kMt} - 1) + M}.$$

For fixed  $M$ ,  $t$ , and  $P_0$  this distance decreases as  $k$  increases; thus, the larger the value of  $k$ , the more rapidly  $P(t)$  approaches  $M$ .

Finally, *numerically*, we tabulate values of  $P(t)$ ,  $t = 0, 0.1, 0.2, \dots, 0.9, 1$ , for the two solutions illustrated graphically above, using  $P_0 = 0.1$  in both cases. Once again the evidence is that the larger value of  $k$  leads to the more rapid approach to  $M$ :

$k = 1$											
$t$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$P(t)$	0.1	0.267	0.695	1.687	3.555	5.999	8.030	9.172	9.679	9.879	9.955
$k = 2$											
$t$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$P(t)$	0.1	0.695	3.555	8.03	9.679	9.955	9.994	9.999	9.999	10.00	10.00

36. With  $x = e^{-50kM}$ ,  $P_0 = 5.308$ ,  $P_1 = 23.192$ , and  $P_2 = 76.212$ , Equation (7) in the text takes the forms

$$\frac{P_0 M}{P_0 + (M - P_0)x} = P_1$$

$$\frac{P_0 M}{P_0 + (M - P_0)x^2} = P_2$$

from which we get

$$P_0 + (M - P_0)x = \frac{P_0 M}{P_1}$$

$$P_0 + (M - P_0)x^2 = \frac{P_0 M}{P_2}$$

or

$$(i) \quad x = \frac{P_0(M - P_1)}{P_1(M - P_0)}$$

$$x^2 = \frac{P_0(M - P_2)}{P_2(M - P_0)}$$

Thus

$$\frac{P_0^2(M - P_1)^2}{P_1^2(M - P_0)^2} = \frac{P_0(M - P_2)}{P_2(M - P_0)}$$

or

$$P_0 P_2 (M - P_1)^2 = P_1^2 (M - P_0)(M - P_2).$$

Expanding gives

$$P_0 P_2 M^2 - 2P_0 P_1 P_2 M + P_0 P_1^2 P_2 = P_1^2 M^2 - P_1^2 (P_0 + P_2)M + P_0 P_1^2 P_2,$$

in which we cancel the final term on each side and solve for

$$(ii) \quad M = \frac{P_1(2P_0 P_2 - P_0 P_1 - P_1 P_2)}{P_0 P_2 - P_1^2}.$$

Substitution of the given values  $P_0 = 5.308$ ,  $P_1 = 23.192$ , and  $P_2 = 76.212$  now gives  $M = 188.121$ . The first equation in (i) and  $x = e^{-kM t_1}$  yield

$$(iii) \quad k = -\frac{1}{M t_1} \ln \frac{P_0(M - P_1)}{P_1(M - P_0)}.$$

Now substitution of  $t_1 = 50$  and our numerical values of  $M$ ,  $P_0$ ,  $P_1$ , and  $P_2$  gives  $k = 0.000167716$ . Finally, substitution of these values of  $k$  and  $M$  (and  $P_0$ ) in the logistic solution (7) gives the logistic model of Eq. (11) in the text.



In Problems 37 and 38 we give just the values of  $k$  and  $M$  calculated using Equations (ii) and (iii) in Problem 36 above, the resulting logistic solution, and the predicted year 2000 population.

37  $k = 0.0000668717$  and  $M = 338.027$ , so that  $P(t) = \frac{25761.7}{76.212 + 261.815e^{-0.0226045t}}$ , which predicts that  $P = 192.525$  in the year 2000.

38.  $k = 0.000146679$  and  $M = 208.250$ , so that  $P(t) = \frac{4829.73}{23.192 + 185.058e^{-0.0305458t}}$ , which predicts that  $P = 248.856$  in the year 2000.

39. Separating variables gives

$$\int \frac{1}{P} dP = \int (k + b \cos 2\pi t) dt, \text{ or}$$

$$\ln P = kt + \frac{b}{2\pi} \sin 2\pi t + C. \text{ The initial}$$

condition  $P(0) = P_0$  implies that  $C = \ln P_0$ , so the desired particular solution is

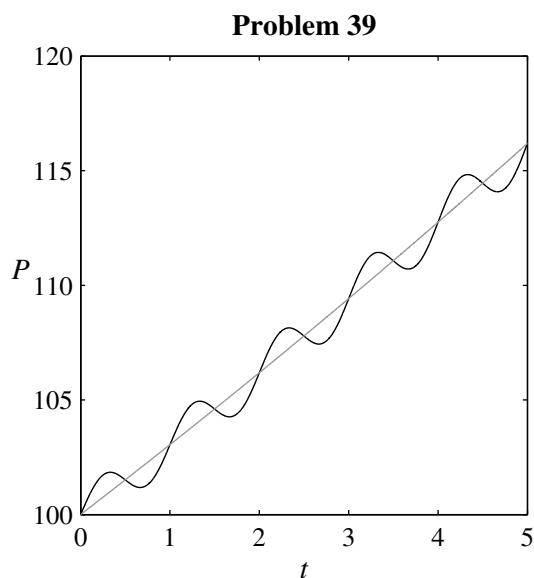
$$P = P_0 \exp\left(kt + \frac{b}{2\pi} \sin 2\pi t\right). \text{ Of course the}$$

natural growth equation  $P' = kP$  with the same initial condition has solution

$$P(t) = P_0 e^{kt}. \text{ The results of both growth}$$

patterns are indicated in the graph shown

with the typical numerical values  $P_0 = 100$ ,  $k = 0.03$ , and  $b = 0.06$ . Under the periodic growth law the population oscillates about the curve representing natural growth. We see that the two agree at the end of each full year.



## SECTION 2.2

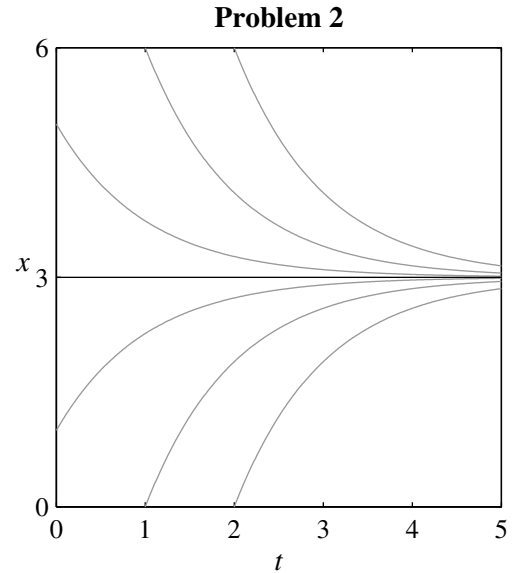
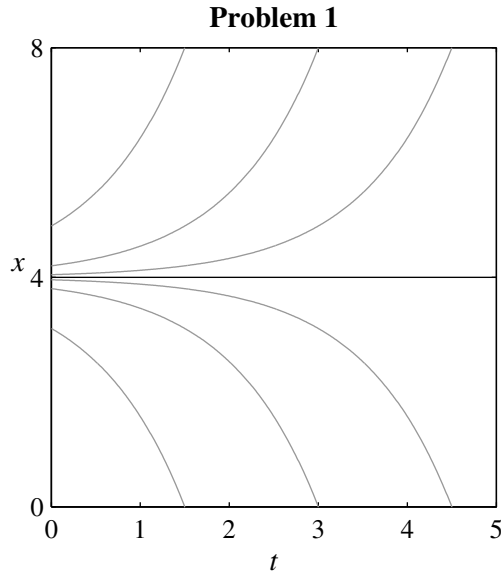
### EQUILIBRIUM SOLUTIONS AND STABILITY

In Problems 1-12 we identify the stable and unstable critical points as well as the funnels and spouts along the equilibrium solutions. In each problem the indicated solution satisfying  $x(0) = x_0$  is derived by separation of variables, and we show typical solution curves corresponding to different values of  $x_0$ .

1. The unstable critical point  $x = 4$  leads to a spout along the equilibrium solution  $x(t) = 4$ .

Separating variables gives  $\int \frac{1}{x-4} dx = \int dt$ , or  $\ln|x-4| = t + C$ , where  $C$  is an arbitrary

constant. Thus the general solution is  $x = Ce^t + 4$ , where  $C$  is an arbitrary nonzero constant. The initial condition  $x(0) = x_0$  then gives  $x_0 = C + 4$ , or  $C = x_0 - 4$ . Thus the solution is given by  $x(t) = (x_0 - 4)e^t + 4$ .



2. The stable critical point  $x = 3$  leads to a funnel along the equilibrium solution  $x(t) = 3$ .

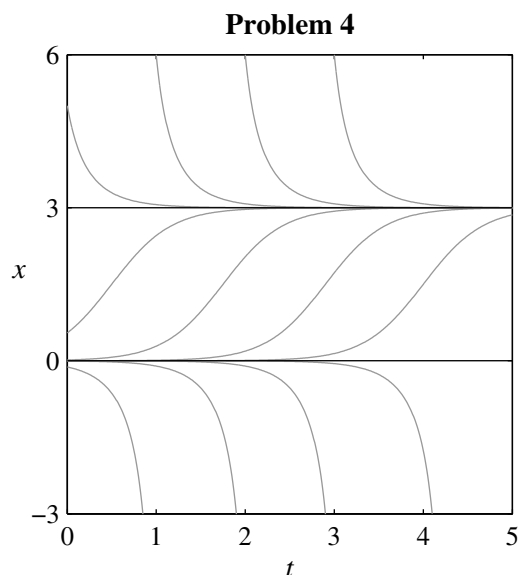
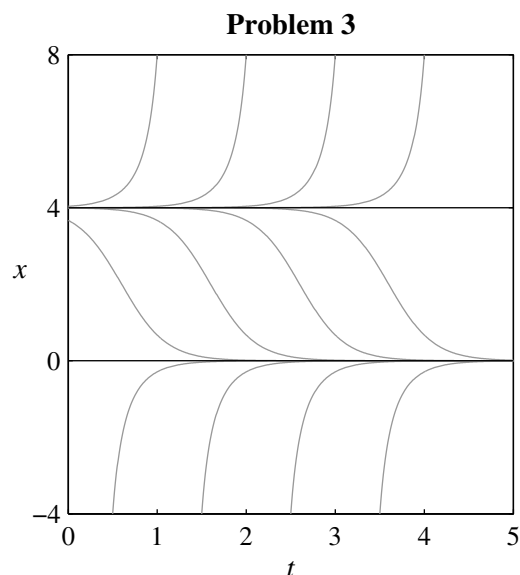
Separating variables gives  $\int \frac{1}{3-x} dx = \int dt$ , or  $\ln|x-3| = -t + C$ , where  $C$  is an arbitrary constant. Thus the general solution is  $x = Ce^{-t} + 3$ , where  $C$  is an arbitrary nonzero constant. The initial condition  $x(0) = x_0$  then gives  $x_0 = C + 3$ , or  $C = x_0 - 3$ . Thus the solution is given by  $x(t) = (x_0 - 3)e^{-t} + 3$ .

3. The stable critical point  $x = 0$  leads to a funnel along the equilibrium solution  $x(t) = 0$ . The unstable critical point  $x = 4$  leads to a spout along the equilibrium solution  $x(t) = 4$ .

Separating variables gives  $\int \frac{1}{x^2 - 4x} dx = \int dt$ , or  $\int \frac{1}{x-4} - \frac{1}{x} dx = \int 4 dt$ . Integrating gives  $\ln|x-4| - \ln|x| = 4t + C$ , or  $\frac{x-4}{x} = Ce^{4t}$ , where  $C$  is an arbitrary nonzero constant.

The initial condition  $x(0) = x_0$  gives  $\frac{x_0 - 4}{x_0} = C$ , leading to  $\frac{x-4}{x} = \frac{x_0 - 4}{x_0} e^{4t}$ , or finally

the solution  $x(t) = \frac{4x_0}{x_0 + (4 - x_0)e^{4t}}$ .



4. The stable critical point  $x = 3$  leads to a funnel along the equilibrium solution  $x(t) = 3$ . The unstable critical point  $x = 0$  leads to a spout along the equilibrium solution  $x(t) = 0$ .

Separating variables gives  $\int \frac{1}{3x-x^2} dx = \int dt$ , or  $\int \frac{1}{x-3} - \frac{1}{x} dx = \int -3dt$ . Integrating gives  $\ln|x-3| - \ln|x| = -3t + C$ , or  $\frac{x-3}{x} = Ce^{-3t}$ , where  $C$  is an arbitrary nonzero

constant. The initial condition  $x(0) = x_0$  gives  $\frac{x_0-3}{x_0} = C$ , leading to  $\frac{x-3}{x} = \frac{x_0-3}{x_0} e^{-3t}$ ,

or finally the solution  $x(t) = \frac{3x_0}{x_0 + (3-x_0)e^{-3t}}$ .

5. The stable critical point  $x = -2$  leads to a funnel along the equilibrium solution  $x(t) = -2$ . The unstable critical point  $x = 2$  leads to a spout along the equilibrium

solution  $x(t) = 2$ . Separating variables gives  $\int \frac{1}{x^2-4} dx = \int dt$ , or

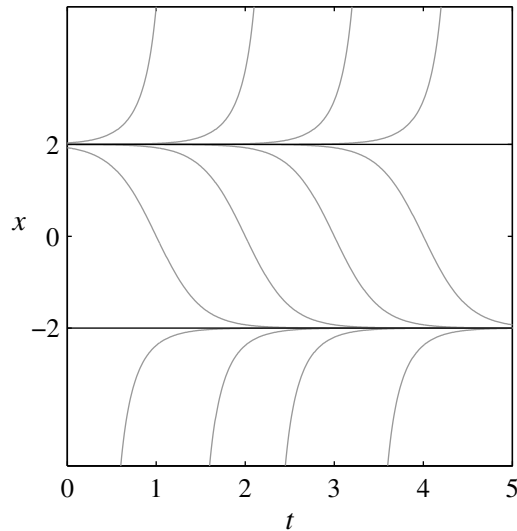
$\int \frac{1}{x-2} - \frac{1}{x+2} dx = \int 4dt$ . Integrating gives  $\ln|x-2| - \ln|x+2| = -4t + C$ , or

$\frac{x-2}{x+2} = Ce^{-4t}$ , where  $C$  is an arbitrary nonzero constant. The initial condition  $x(0) = x_0$

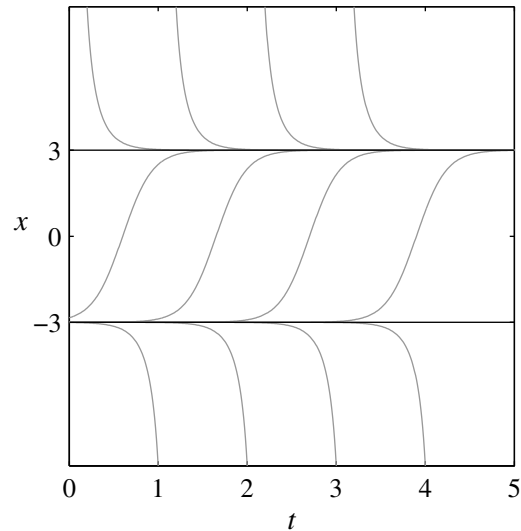
gives  $\frac{x_0-2}{x_0+2} = C$ , leading to  $\frac{x-2}{x+2} = \frac{x_0-2}{x_0+2} e^{-4t}$ , or finally the solution

$x(t) = 2 \frac{(x_0+2) + (x_0-2)e^{-4t}}{(x_0+2) - (x_0-2)e^{-4t}}$ .

Problem 5



Problem 6



6. The stable critical point  $x = 3$  leads to a funnel along the equilibrium solution  $x(t) = 3$ . The unstable critical point  $x = -3$  leads to a spout along the equilibrium solution

$$x(t) = -3. \text{ Separating variables gives } \int \frac{1}{9-x^2} dx = \int dt, \text{ or } \int \frac{1}{x+3} - \frac{1}{x-3} dx = \int 6 dt.$$

Integrating gives  $\ln|x+3| - \ln|x-3| = 6t + C$ , or  $\frac{x+3}{x-3} = Ce^{6t}$ , where  $C$  is an arbitrary

nonzero constant. The initial condition  $x(0) = x_0$  gives  $\frac{x_0+3}{x_0-3} = C$ , leading to

$$\frac{x+3}{x-3} = \frac{x_0+3}{x_0-3} e^{-3t}, \text{ or finally the solution } x(t) = 3 \frac{(x_0-3) + (x_0+3)e^{6t}}{(3-x_0) + (x_0+3)e^{6t}}.$$

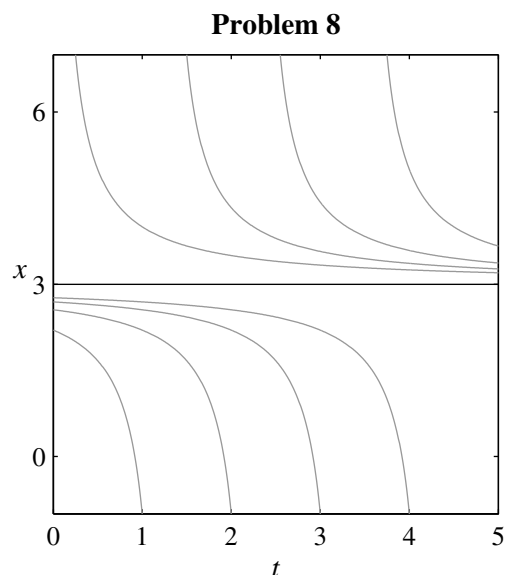
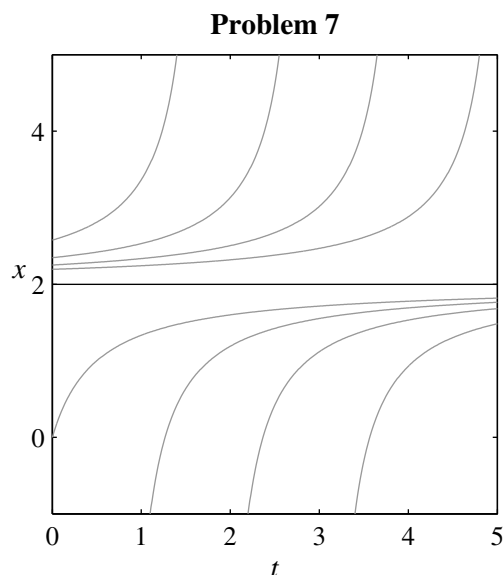
7. The lone critical point  $x = 2$  is *semi-stable*; solutions with  $x_0 > 2$  approach  $+\infty$  as  $t$  increases, whereas those with  $x_0 < 2$  approach 2 as  $t$  increases. Separating variables

$$\text{gives } \int \frac{1}{(x-2)^2} dx = \int dt, \text{ or } \frac{1}{x-2} = -t + C, \text{ where } C \text{ is an arbitrary nonzero constant.}$$

The initial condition  $x(0) = x_0$  gives  $\frac{1}{x_0-2} = C$ , leading to

$$\frac{1}{x-2} = -t + \frac{1}{x_0-2} = \frac{1-t(x_0-2)}{x_0-2}, \text{ or finally the solution}$$

$$x(t) = 2 + \frac{x_0-2}{1-t(x_0-2)} = \frac{x_0(2t-1) - 4t}{(x_0-2)t-1}.$$



8. The lone critical point  $x = 3$  is *semi-stable*; solutions with  $x_0 < 3$  approach  $-\infty$  as  $t$  increases, whereas those with  $x_0 > 3$  approach 3 as  $t$  increases. Separating variables gives  $\int \frac{1}{(x-3)^2} dx = -\int dt$ , or  $\frac{1}{x-3} = t + C$ , where  $C$  is an arbitrary nonzero constant.

The initial condition  $x(0) = x_0$  gives  $C = \frac{1}{x_0 - 3}$ , leading to

$$\frac{1}{x-3} = t + \frac{1}{x_0 - 3} = \frac{1 + t(x_0 - 3)}{x_0 - 3}, \text{ or finally the solution}$$

$$x(t) = 3 + \frac{x_0 - 3}{1 + t(x_0 - 3)} = \frac{x_0(3t + 1) - 9t}{(x_0 - 3)t + 1}.$$

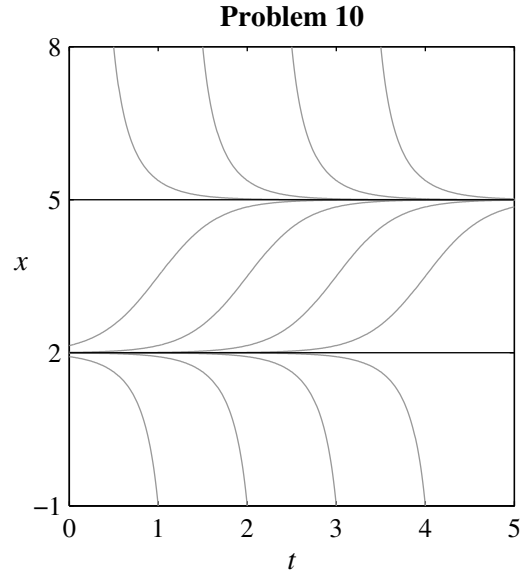
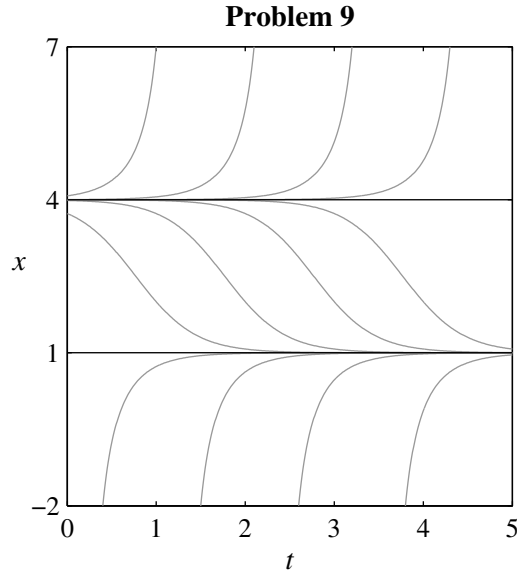
9. Factoring gives  $x^2 - 5x + 4 = (x - 4)(x - 1)$ . The stable critical point  $x = 1$  leads to a funnel along the equilibrium solution  $x(t) = 1$ . The unstable critical point  $x = 4$  leads to a spout along the equilibrium solution  $x(t) = 4$ . Separating variables gives

$$\int \frac{1}{(x-4)(x-1)} dx = \int dt, \text{ or } \int \frac{1}{x-4} - \frac{1}{x-1} dx = \int 3 dt. \text{ Integrating gives}$$

$$\ln|x-4| - \ln|x-1| = 3t + C, \text{ or } \frac{x-4}{x-1} = Ce^{3t}, \text{ where } C \text{ is an arbitrary nonzero constant.}$$

The initial condition  $x(0) = x_0$  gives  $\frac{x_0 - 4}{x_0 - 1} = C$ , leading to  $\frac{x-4}{x-1} = \frac{x_0 - 4}{x_0 - 1} e^{3t}$ , or finally

$$\text{the solution } x(t) = \frac{4(1-x_0) + (x_0-4)e^{3t}}{(1-x_0) + (x_0-4)e^{3t}}.$$



- 10.** Factoring gives  $7x - x^2 - 10 = -(x-5)(x-2)$ . The stable critical point  $x = 5$  leads to a funnel along the equilibrium solution  $x(t) = 5$ . The unstable critical point  $x = 2$  leads to a spout along the equilibrium solution  $x(t) = 4$ . Separating variables gives

$$\int \frac{-1}{(x-5)(x-2)} dx = \int dt, \text{ or } \int \frac{1}{x-2} - \frac{1}{x-5} dx = \int 3 dt. \text{ Integrating gives}$$

$$\ln|x-2| - \ln|x-5| = 3t + C, \text{ or } \frac{x-2}{x-5} = Ce^{3t}, \text{ where } C \text{ is an arbitrary nonzero constant.}$$

$$\text{The initial condition } x(0) = x_0 \text{ gives } \frac{x_0-2}{x_0-5} = C, \text{ leading to } \frac{x-2}{x-5} = \frac{x_0-2}{x_0-5} e^{3t}, \text{ or finally}$$

$$\text{the solution } x(t) = \frac{2(5-x_0) + 5(x_0-2)e^{3t}}{(5-x_0) + (x_0-2)e^{3t}}.$$

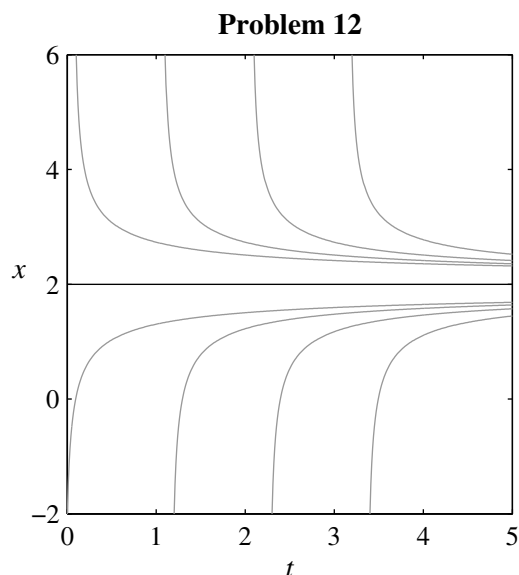
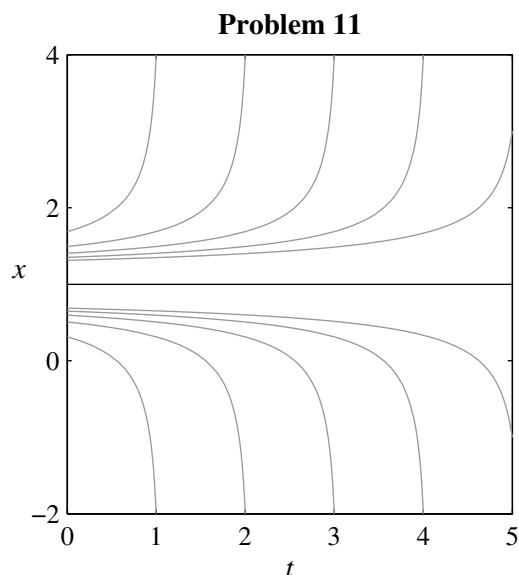
- 11.** The unstable critical point  $x = 1$  leads to a spout along the equilibrium solution  $x(t) = 4$ .

$$\text{Separating variables gives } \int \frac{1}{(x-1)^3} dx = \int dt, \text{ and integrating gives } \frac{1}{2(x-1)^2} = -t + C,$$

$$\text{where } C \text{ is an arbitrary constant. The initial condition } x(0) = x_0 \text{ gives } \frac{1}{2(x_0-1)^2} = C,$$

$$\text{leading to } \frac{1}{2(x-1)^2} = -t + \frac{1}{2(x_0-1)^2}, \text{ or } (x-1)^2 = \frac{(x_0-1)^2}{1-2t(x_0-1)^2}, \text{ or finally the solution}$$

$$x(t) = 1 \pm \frac{x_0-1}{\sqrt{1-2t(x_0-1)^2}}.$$



- 12.** The stable critical point  $x = 2$  leads to a funnel along the equilibrium solution  $x(t) = 2$ .

Separating variables gives  $\int \frac{1}{(2-x)^3} dx = \int dt$ , and integrating gives  $\frac{1}{2(2-x)^2} = t + C$ ,

where  $C$  is an arbitrary constant. The initial condition  $x(0) = x_0$  gives  $\frac{1}{2(2-x_0)^2} = C$ ,

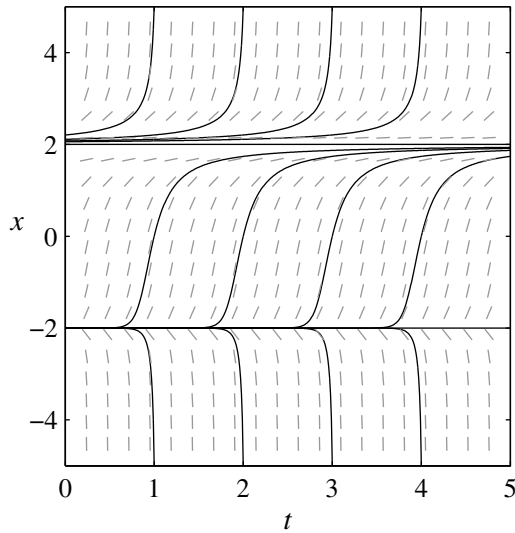
leading to  $\frac{1}{2(2-x)^2} = t + \frac{1}{2(2-x_0)^2}$ , or  $(2-x)^2 = \frac{(2-x_0)^2}{2t(2-x_0)^2 + 1}$ , or finally the

solution  $x(t) = 2 \pm \frac{2-x_0}{\sqrt{2t(2-x_0)^2 + 1}}$ .

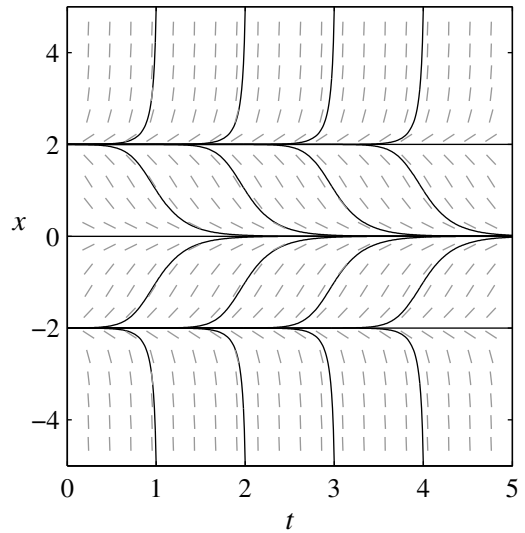
In each of Problems 13–18 we present the figure showing the slope field and typical solution curves, and then record the visually apparent classification of critical points for the given differential equation.

- 13.** The critical points  $x = 2$  and  $x = -2$  are both unstable.

**Problem 13**

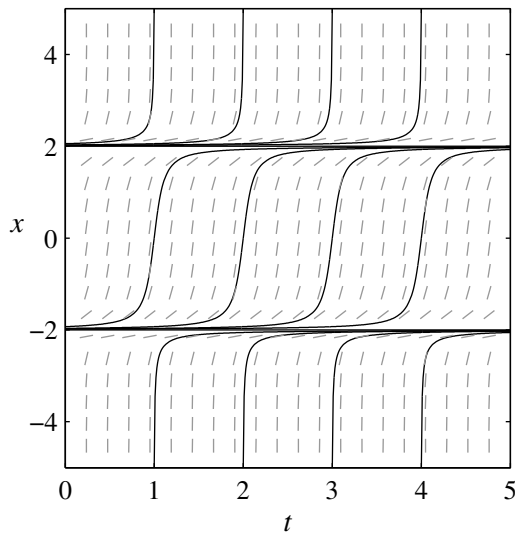


**Problem 14**

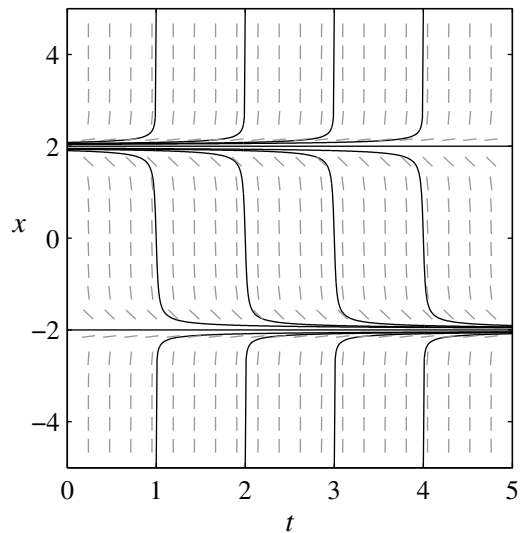


14. The critical points  $x = \pm 2$  are both unstable, whereas the critical point  $x = 0$  is stable.
15. The critical points  $x = 2$  and  $x = -2$  are both unstable.

**Problem 15**

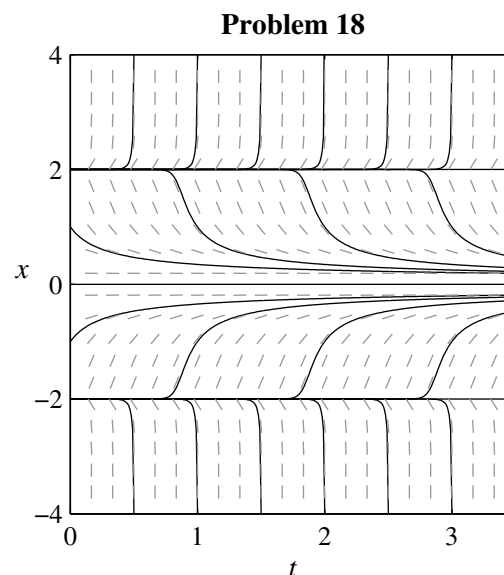
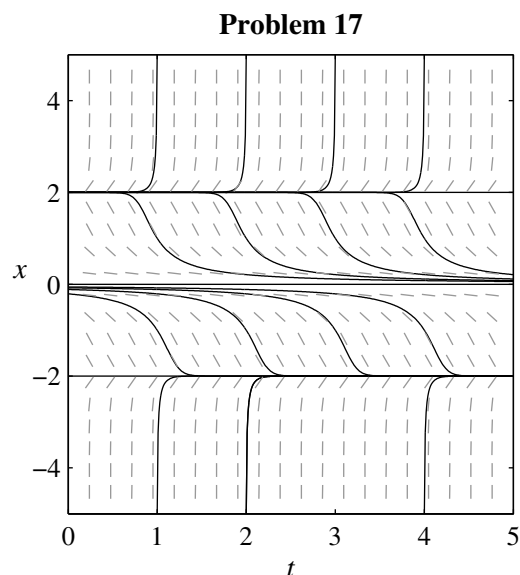


**Problem 16**



16. The critical point  $x = 2$  is unstable, while the critical point  $x = -2$  is stable.
17. The critical points  $x = 2$  and  $x = 0$  are unstable, while the critical point  $x = -2$  is stable.





18. The critical points  $x = 2$  and  $x = -2$  are unstable, whereas the critical point  $x = 0$  is stable.
19. The critical points of the given differential equation are the roots of the quadratic equation  $\frac{1}{10}x(10-x) - h = 0$ , that is,  $x^2 - 10x + 10h = 0$ . Thus a critical point  $c$  is given in terms of  $h$  by

$$c = \frac{10 \pm \sqrt{100 - 40h}}{2} = 5 \pm \sqrt{25 - 10h}.$$

It follows that there is no critical point if  $h > \frac{5}{2}$ , only the single critical point  $c = 0$  if  $h = \frac{5}{2}$ , and two distinct critical points if  $h < \frac{5}{2}$ , so that  $10 - 25h > 0$ . Hence the bifurcation diagram in the  $hc$ -plane is the parabola  $(c - 5)^2 = 25 - 10h$  that is obtained upon squaring to eliminate the square root above.

20. The critical points of the given differential equation are the roots of the quadratic equation  $\frac{1}{100}x(x-5) + s = 0$ , that is,  $x^2 - 5x + 100s = 0$ . Thus a critical point  $c$  is given in terms of  $s$  by

$$c = \frac{5 \pm \sqrt{25 - 400s}}{2} = \frac{5}{2} \pm \frac{5}{2}\sqrt{1 - 16s}.$$

It follows that there is no critical point if  $s > \frac{1}{16}$ , only the single critical point  $c = \frac{5}{2}$  if  $s = \frac{1}{16}$ , and two distinct critical points if  $s < \frac{1}{16}$ , so that  $1 - 16s > 0$ . Hence the bifurcation diagram in the  $sc$ -plane is the parabola  $(2c - 5)^2 = 25(1 - 16s)$  that is obtained upon elimination of the radical above.

21. (a) If  $k = -a^2$ , where  $a \geq 0$ , then  $kx - x^3 = -a^2x - x^3 = -x(a^2 + x^2) = 0$  only if  $x = 0$ , so the only critical point is  $c = 0$ . If  $a > 0$ , then we can solve the differential equation by writing

$$\int \frac{a^2}{x(a^2 + x^2)} dx = \int \frac{1}{x} - \frac{x}{a^2 + x^2} dx = -\int a^2 dt,$$

or  $\ln x - \frac{1}{2} \ln(a^2 + x^2) = -a^2t + C$ , or  $\frac{x^2}{a^2 + x^2} = Ce^{-2a^2t}$ , where  $C$  is an arbitrary nonzero constant. Solving for  $x^2$  gives  $x^2 = \frac{a^2Ce^{-2a^2t}}{1 - Ce^{-2a^2t}}$ , from it follows that  $x \rightarrow 0$  as  $t \rightarrow \infty$ , so the critical point  $c = 0$  is *stable*.

(b) If  $k = a^2$ , where  $a > 0$ , then  $kx - x^3 = +a^2x - x^3 = -x(x + a)(x - a) = 0$  if either  $x = 0$  or  $x = \pm a = \pm\sqrt{k}$ . Thus we have the three critical points  $c = 0$  and  $c = \pm\sqrt{k}$ ; this observation, together with part (a), yields the pitchfork bifurcation diagram shown in Fig. 2.2.13 of the textbook. If  $x(0) \neq 0$ , then we can solve the differential equation by writing

$$\int \frac{2a^2}{x(x-a)(x+a)} dx = \int -\frac{2}{x} + \frac{1}{x-a} + \frac{1}{x+a} dx = -\int 2a^2 dt,$$

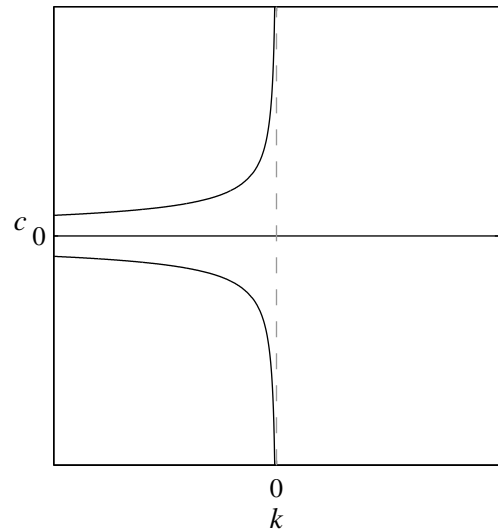
or  $-2 \ln x + \ln(x-a) + \ln(x+a) = -2a^2t$ , or  $\frac{x^2 - a^2}{x^2} = Ce^{-2a^2t}$ , where  $C$  is an arbitrary nonzero constant. Solving for  $x^2$  gives  $x^2 = \frac{a^2}{1 - Ce^{-2a^2t}}$ , and so  $x = \frac{\pm\sqrt{k}}{\sqrt{1 - Ce^{-2a^2t}}}$ . It follows that if  $x(0) \neq 0$ , then  $x \rightarrow \sqrt{k}$  if  $x > 0$  and  $x \rightarrow -\sqrt{k}$  if  $x < 0$ . This implies that the critical point  $c = 0$  is *unstable*, while the critical points  $c = \pm\sqrt{k}$  are *stable*.

22. If  $k = 0$ , then the only critical point  $c = 0$  of the equation  $x' = x$  is unstable, because the solutions  $x(t) = x_0e^t$  diverge to infinity if  $x_0 \neq 0$ . If  $k = +a^2 > 0$ , then  $x + a^2x^3 = x(1 + a^2x^2) = 0$  only if  $x = 0$ , so again  $c = 0$  is the only critical point. If  $k = -a^2 < 0$ , then  $x - a^2x^3 = x(1 - a^2x^2) = x(1 - ax)(1 + ax) = 0$  if either  $x = 0$  or

$x = \pm \frac{1}{a} = \pm \sqrt{\frac{-1}{k}}$ . Hence the bifurcation

diagram of the differential equation  $x' = x + kx^3$  looks as shown.

**Problem 22**



23. (a) If  $h < kM$ , then writing the differential equation as

$$x' = kx(M - x) - hx = kx \left[ \left( M - \frac{h}{k} \right) - x \right],$$

still a logistic equation but with the *reduced* limiting population  $M - \frac{h}{k}$ .

- (b) If  $h > kM$ , then the differential equation can be rewritten in the form  $x' = -ax - bx^2$ , with  $a$

and  $b$  both positive. The solution of this equation is  $x(t) = \frac{ax_0}{(a + bx_0)e^{at} - bx_0}$ , so it is clear that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

24. Separating variables gives  $\int \frac{1}{(N-x)(x-H)} dx = \int k dt$ . By the method of partial fractions,

$$\int \frac{1}{(N-x)(x-H)} dx = \frac{1}{N-H} \int \frac{1}{N-x} + \frac{1}{x-H} dx = \frac{1}{N-H} \ln \left| \frac{x-H}{x-N} \right|,$$

and so the general solution of the differential equation is given by

$$\frac{1}{N-H} \ln \left| \frac{x-H}{x-N} \right| = kt + C, \text{ or } \frac{x-H}{x-N} = Ce^{k(N-H)t}, \text{ where } C \text{ is an arbitrary nonzero}$$

constant. The initial condition  $x(0) = x_0$  gives  $C = \frac{x_0 - H}{x_0 - N}$ , so that

$$\frac{x-H}{x-N} = \frac{x_0 - H}{x_0 - N} e^{k(N-H)t}, \text{ and solving for } x \text{ leads to the solution}$$

$$x(t) = \frac{N(x_0 - H) - H(x_0 - N)e^{-k(N-H)t}}{(x_0 - H) - (x_0 - N)e^{-k(N-H)t}}$$

in the text.

25. In the first alternative form that is given, all of the coefficients within parentheses are positive if  $H < x_0 < N$ . Hence it is clear that  $x(t) \rightarrow N$  as  $t \rightarrow \infty$ , which confirms (17).

In the second alternative form, all of the coefficients within parentheses are positive if  $x_0 < H$ . Hence the denominator is initially equal to  $N - H > 0$ , but decreases as  $t$

increases, and reaches the value 0 when  $t = t_1 = \frac{1}{k(N-H)} \ln \frac{N-x_0}{H-x_0} > 0$ . Meanwhile the numerator is initially  $(N-H)x_0$ , but approaches  $(H-N)(H-x_0) < 0$  as  $t \rightarrow t_1$ . Conclusion (18) follows.

26. If  $4h = kM^2$ , then Equations (13) and (14) in the text show that the differential equation takes the form  $x' = -k\left(\frac{M}{2} - x\right)^2$  with the single critical point  $x = \frac{M}{2}$ . This equation is readily solved by separation of variables, but clearly  $x'$  is negative whether  $x$  is less than or greater than  $\frac{M}{2}$ .

27. Separation of variables in the differential equation  $x' = -k[(x-a)^2 + b^2]$  yields

$$x(t) = a - b \tan\left(bkt + \tan^{-1} \frac{a-x_0}{b}\right).$$

It follows that  $x(t) \rightarrow -\infty$  in a finite period of time.

28. Aside from a change in sign, this calculation is the same as that indicated in Equations (13) and (14) in the text.
29. This is simply a matter of analyzing the signs of  $x'$  in the various cases  $x < a$ ,  $a < x < b$ ,  $b < x < c$ , and  $c > x$ . Alternatively, plot slope fields and typical solution curves for the two differential equations using typical numerical values such as  $a = -1$ ,  $b = 1$ , and  $c = 2$ .

## SECTION 2.3

### ACCELERATION-VELOCITY MODELS

This section consists of three essentially independent subsections that can be studied separately: resistance proportional to velocity, resistance proportional to velocity-squared, and inverse-square gravitational acceleration.

1. The velocity  $v$  of the car (in km/hr) is related to the time  $t$  (in seconds) by the initial value problem  $v' = k(250 - v)$ ,  $v(0) = 0$ ,  $v(10) = 100$ . Separating variables gives

$$\int \frac{1}{250-v} dv = \int k dt, \text{ and integration yields } \ln|250-v| = kt + C, \text{ or } 250-v = Ce^{kt}, \text{ or}$$

finally  $v = Ce^{kt} + 250$ , where  $C$  is an arbitrary nonzero constant. The initial condition  $v(0) = 0$  gives  $C = -250$ , so that  $v = 250(1 - e^{kt})$ , and the condition  $v(10) = 100$

implies that  $k = \frac{1}{10} \ln\left(\frac{250}{150}\right) \approx 0.0511$ . Finally, solving the equation  $v(t) = 200$  for  $t$  gives  $t = -\frac{\ln 50}{250k} \approx 31.5$  sec.

2. (a) The solution of the initial value equation  $\frac{dv}{dt} = -kv$ ,  $v(0) = v_0$  is  $v(t) = v_0 e^{-kt}$ . Then  $x(t) = \int v(t) dt = \int v_0 e^{-kt} dt = -\frac{v_0}{k} e^{-kt} + C$ , and the initial condition  $x(0) = x_0$  gives  $C = x_0 + \frac{v_0}{k}$ . Thus  $x(t) = -\frac{v_0}{k} e^{-kt} + x_0 + \frac{v_0}{k} = x_0 + \frac{v_0}{k} (1 - e^{-kt})$ .
- (b)  $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} \left[ x_0 + \frac{v_0}{k} (1 - e^{-kt}) \right] = x_0 + \frac{v_0}{k}$ . (Instructors may wish to explore how this expression changes with  $v_0$  and  $k$ , and what that means in the context of the problem.)
3. The velocity  $v$  of the boat (in ft/s) is related to the time  $t$  (in seconds) by the initial value problem  $v' = -kv$ ,  $v(0) = 40$ ,  $v(10) = 20$ . By Problem 2a,  $v(t) = 40^{-kt}$ , and the condition  $v(10) = 20$  implies that  $k = \frac{1}{10} \ln 2 \approx 0.0693$ . By Problem 2b, then, the boat travels a distance of  $\frac{v_0}{k} = \frac{40 \cdot 10}{\ln 2} \approx 577$  ft altogether.
4. Separating variables gives  $\int \frac{1}{v^2} dv = -\int k dt$ , or  $-\frac{1}{v} = -kt + C$ . The initial condition  $v(0) = v_0$  gives  $C = -\frac{1}{v_0}$ , so that  $-\frac{1}{v} = -kt - \frac{1}{v_0}$ , or  $v = \frac{1}{kt + \frac{1}{v_0}} = \frac{v_0}{1 + v_0 kt}$ . Then
- $$x(t) = \int v(t) dt = \int \frac{v_0}{1 + v_0 kt} dt = \frac{1}{k} \ln(1 + v_0 kt) + C'$$
- and the initial condition  $x(0) = x_0$  implies that  $C' = x_0$ . Thus  $x(t) = \frac{1}{k} \ln(1 + v_0 kt) + x_0$ , so that  $x(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . The reason for the difference in outcomes between Problems 2 and 4 is that  $v^2$  is much smaller than  $v$  when  $v$  is small, so that at low speed the medium in Problem 4 offers less resistance than the one in Problem 2.
5. We are assuming that the velocity  $v$  of the motorboat satisfies the initial value problem  $v' = -kv^2$ ,  $v(0) = 40$ , with  $v(10) = 20$  as well. We seek  $x(60)$ . The result of Problem

4 gives  $v = \frac{v_0}{1 + v_0 kt} = \frac{40}{1 + 40kt}$ , and then the condition  $v(10) = 20$  implies that

$$20 = \frac{40}{1 + 400k}, \text{ or } k = \frac{1}{400}. \text{ Thus } v(t) = \frac{400}{10 + t}, \text{ and then}$$

$$x = \int v(t) dt = 400 \ln(t + 10) + C.$$

The initial condition  $x(0) = 0$  implies that  $C = -400 \ln 10$ , so that  $x(t) = 400 \ln \frac{t + 10}{10}$ . It follows that  $x(60) = 400 \ln 7 \approx 778$  ft.

6. Separating variables gives  $\int \frac{1}{v^{3/2}} dv = -\int k dt$ , or  $-2v^{-1/2} = -kt + C$ . The initial condition  $v(0) = v_0$  gives  $C = -\frac{2}{\sqrt{v_0}}$ , so that  $\frac{2}{\sqrt{v}} = kt + \frac{2}{\sqrt{v_0}}$ , or

$$v(t) = \left( \frac{2}{kt + \frac{2}{\sqrt{v_0}}} \right)^2 = \frac{4v_0}{(\sqrt{v_0}kt + 2)^2}.$$

Then

$$x = \int \frac{4v_0}{(\sqrt{v_0}kt + 2)^2} dt = -\frac{4\sqrt{v_0}}{k(\sqrt{v_0}kt + 2)} + C,$$

and the initial condition  $x(0) = x_0$  yields  $x_0 = -\frac{2\sqrt{v_0}}{k} + C$ , or  $C = x_0 + \frac{2\sqrt{v_0}}{k}$ , so that

$$x(t) = -\frac{4\sqrt{v_0}}{k(\sqrt{v_0}kt + 2)} + x_0 + \frac{2\sqrt{v_0}}{k} = x_0 + \frac{2\sqrt{v_0}}{k} \left( 1 - \frac{2}{\sqrt{v_0}kt + 2} \right).$$

Finally, letting  $t \rightarrow \infty$  gives  $x(t) \rightarrow x_0 + \frac{2\sqrt{v_0}}{k}$ , indeed a finite distance.

7. The car satisfies the initial value problem  $v' = 10 - 0.1v$ ,  $v(0) = 0$ . Separating variables gives  $\int \frac{1}{10 - 0.1v} dv = \int dt$ , or  $\ln(10 - 0.1v) = -\frac{t}{10} + C$ . The initial condition  $v(0) = 0$  gives  $C = \ln 10$ , so that  $\ln(10 - 0.1v) = -\frac{t}{10} + \ln 10$ , or  $\ln(1 - 0.01v) = -\frac{t}{10}$ , or  $v(t) = 100(1 - e^{-t/10})$ . As  $t \rightarrow \infty$ , we find  $v(t) \rightarrow 100$  ft/sec, the answer to **a**. Further, setting  $v(t) = 90$  ft/sec (that is, 90% of limiting velocity) gives  $t = 10 \ln 10 \approx 23.0259$  sec.

Since  $x = \int 100(1 - e^{-t/10}) dt = 100t + 1000e^{-t/10} + C'$ , where the initial condition  $x(0) = 0$  gives  $C' = -1000$ , we find that  $x(23.0259) \approx 1402.59$  ft, the answer to **b**.

8. The car now satisfies the initial value problem  $v' = 10 - 0.001v^2$ ,  $v(0) = 0$ . Separating variables gives  $\int \frac{1}{10 - 0.001v^2} dv = \int dt$ . By the antiderivative formula

$$\int \frac{1}{1 - u^2} du = \tanh^{-1} u + C, \text{ the general solution is } \tanh^{-1} \frac{v}{100} = \frac{t}{10} + C, \text{ and then the}$$

initial condition  $v(0) = 0$  implies that  $C = 0$ . Thus  $v(t) = 100 \tanh \frac{t}{10} = 100 \frac{e^{t/10} - e^{-t/10}}{e^{t/10} + e^{-t/10}}$ ,

which approaches 100 ft/sec as  $t \rightarrow \infty$ , the answer to **a**, identical to that in the preceding

Problem. However in this case setting  $v(t) = 90$  ft/sec gives  $\frac{e^{t/5} - 1}{e^{t/5} + 1} = 0.9$ , or

$$t = 5 \ln 19 \approx 14.7222 \text{ sec. Since } x = \int 100 \tanh \frac{t}{10} dt = 1000 \ln \cosh \frac{t}{10} + C', \text{ where the}$$

initial condition  $x(0) = 0$  gives  $C' = 0$ , we find that  $x(14.7222) \approx 830.366$  ft, the answer to **b**. Thus the car achieves 90% of “top end” both in a shorter time and over a shorter distance.

9. Separating variables gives  $\int \frac{1000}{5000 - 100v} dv = \int dt$ , or  $10 \ln(5000 - 100v) = -t + C$ , or  $v = 50 + Ce^{-t/10}$ . The initial condition  $v(0) = 0$  implies that or  $C = -50$ , so that  $v(t) = 50(1 - e^{-t/10})$ . As  $t \rightarrow \infty$ ,  $v(t) \rightarrow 50$  ft/sec  $\approx 34$  mph.

10. We need to solve two initial value problems in succession.

Over the first 20 seconds the woman's velocity  $v(t)$  satisfies the initial value problem

$$v' = -32 - 0.15v, v(0) = 0. \text{ Separating variables gives } \int \frac{1}{32 + 0.15v} dv = -\int dt, \text{ or}$$

$$\ln(32 + 0.15v) = -0.15t + C, \text{ or } v = \frac{1}{0.15}(Ce^{-0.15t} - 32). \text{ The initial condition } v(0) = 0$$

gives  $C = 32$ , so that  $v(t) = \frac{32}{0.15}(e^{-0.15t} - 1) \approx 213.333(e^{-0.15t} - 1)$ . Thus when her

parachute opens her velocity is  $v(20) \approx -202.712$  ft/sec. Her altitude  $y(t)$  satisfies

$$y = \int 213.333(e^{-0.15t} - 1) dt \approx -1422.22e^{-0.15t} - 213.333t + C',$$

where the initial condition  $y(0) = 10,000$  gives  $C' = 11422.22$ . So her altitude is  $y(20) \approx 7084.75$  ft at the point when her parachute opens.

After the parachute opens,  $v(t)$  satisfies the initial value problem  $v' = -32 - 1.5v$ ,  $v(0) \approx -202.712$ , where we reset time so that  $t = 0$  when the parachute opens. Solving as above gives  $v(t) = -21.3333 - 181.379e^{-1.5t}$ , followed by  $y(t) = 6964.83 + 120.919e^{-1.5t} - 21.3333t$ . Solving the equation  $y(t) = 0$  using technology yields  $t = 326.476$  sec.

All told, the woman's total time of descent is  $20 + 326.476 = 346.476$  sec, about 5 minutes and 46.5 seconds. Moreover, her impact speed is 21.33 ft/sec, about 15 mph.

11. If the paratrooper's terminal velocity was  $100 \text{ mph} = \frac{440}{3} \text{ ft/sec}$ , then Equation (7) in the text yields  $\frac{g}{\rho} = \frac{440}{3}$ , or  $\rho = \frac{3}{440} \cdot 32 = \frac{12}{55}$ . Equation (9) then becomes

$$y(t) = -1200 + \frac{440}{3}t - \frac{55}{12} \cdot \frac{440}{3} (1 - e^{-12t/55}),$$

and solving the equation  $y(t) = 0$  via technology gives  $t \approx 12.5$  sec. Thus the newspaper account is inaccurate.

12. The mass of the drums is given by  $m = \frac{W}{g} = \frac{640}{32} = 20$  slugs. With  $B = 62.5 \cdot 8 = 500$  lbs and  $F_R = -v$  lbs, the force equation becomes

$$20 \frac{dv}{dt} = -640 + 500 - v = -140 - v.$$

Its solution with  $v(0) = 0$  is  $v(t) = 140(e^{-0.05t} - 1)$ , and then integration with  $y(0) = 0$  yields  $y(t) = 2800(e^{-0.05t} - 1) - 140t$ . Using these equations we find that

$t = 20 \ln \frac{28}{13} \approx 15.35$  sec when  $v = 75$  ft/sec, and that  $y(15.35) \approx 648.31$  ft. Thus the maximum safe depth is just under 650 ft.

Given the hints and integrals provided in the text, Problems 13–16 are fairly straightforward (and fairly tedious) integration problems.

17. Equation (13) from the text gives

$$v(t) = \sqrt{\frac{9.8}{0.0011}} \tan(C_1 - t\sqrt{0.0011 \cdot 9.8}) = 94.3880 \tan(C_1 - 0.1038267t),$$

where  $C_1 = \tan^{-1}\left(49\sqrt{\frac{0.0011}{9.8}}\right) \approx 0.4788372$ . Thus

$$v(t) = 94.3880 \tan(0.4788372 - 0.1038267t).$$



Then Equation (14) gives

$$y(t) = \frac{1}{0.0011} \ln \left| \frac{\cos(0.4788372 - \sqrt{0.0011 \cdot 9.8}t)}{\cos 0.4788372} \right| \approx 909.0909 \ln \left| \frac{\cos(0.4788372 - 0.103827t)}{\cos 0.4788372} \right|.$$

Setting  $v(t) = 0$  leads to  $t \approx 4.612$  sec, at which time  $y \approx 108.465$  m.

18. We solve the initial value problem  $v' = -9.8 + 0.0011v^2$ ,  $v(0) = 0$  much as in Problem 17, except using hyperbolic rather than ordinary trigonometric functions. We first get

$$v(t) = -94.3841 \tanh(0.103827t),$$

and then integration with  $y(0) = 108.47$  gives

$$y(t) = 108.47 - 909.052 \ln[\cosh(0.103827t)].$$

We solve  $y(0) = 0$  for  $t = \frac{1}{0.103827} \cosh^{-1} \left[ \exp\left(\frac{108.47}{909.052}\right) \right] \approx 4.7992$ , and then calculate  $v(4.7992) = -43.489$ .

19. The initial value problem for the velocity of the motorboat is  $v' = 4 - \frac{1}{400}v^2$ ,  $v(0) = 0$ .

Separating variables gives  $\int \frac{1}{4 - \frac{1}{400}v^2} dv = \int dt$ , or  $\int \frac{1/40}{1 - (v/40)^2} dv = \int \frac{1}{10} dt$ , or

$\tanh^{-1} \frac{v}{40} = \frac{t}{10} + C$ . The initial condition  $v(0) = 0$  gives  $C = 0$ , so that

$v(t) = 40 \tanh \frac{t}{10}$ . Finally,  $v(10) = \tanh 1 \approx 30.46$  ft/sec and

$\lim_{t \rightarrow \infty} v(t) = 40 \lim_{t \rightarrow \infty} \tanh \frac{t}{10} = 40$  ft/sec.

20. The initial value problem for the velocity of the arrow is  $v' = -32 - \frac{1}{800}v^2$ ,  $v(0) = 160$ , with the added condition that  $y(0) = 0$ , where  $y$  is the height of the arrow. Separating

variables gives  $\int \frac{1}{32 + \frac{1}{800}v^2} dv = -\int dt$ , or  $\int \frac{1/160}{1 - (v/160)^2} dv = -\int \frac{1}{5} dt$ , or

$\tan^{-1} \frac{v}{160} = -\frac{t}{5} + C$ . The initial condition  $v(0) = 160$  implies that  $C = \frac{\pi}{4}$ , and so

$v(t) = \frac{dy}{dt} = 160 \tan\left(\frac{\pi}{4} - \frac{t}{5}\right)$ . Integrating then gives  $y(t) = 800 \ln \left[ \cos\left(\frac{\pi}{4} - \frac{t}{5}\right) \right] + C'$ ,

and then the condition  $y(0) = 0$  gives  $0 = 800 \ln \frac{\sqrt{2}}{2} + C'$ , or  $C' = 400 \ln 2$ . Altogether  $y(t) = 800 \ln \left[ \cos \left( \frac{\pi}{4} - \frac{t}{5} \right) \right] + 400 \ln 2$ . Solving  $v(t) = 0$  gives  $t = 3.92699$ , and then calculation gives  $y(3.92699) = 277.26$  ft.

21. The initial value problem for the velocity of the ball is  $v' = -g - \rho v^2$ ,  $v(0) = v_0$ , with the added condition that  $y(0) = 0$ , where  $y$  is the height of the ball. Separating variables

gives  $\int \frac{dv}{g + \rho v^2} = -\int dt$ , or  $\int \frac{\sqrt{\rho/g}}{1 + (\sqrt{\rho/g} v)^2} dv = -\int \sqrt{g\rho} dt$ , or

$\tan^{-1}(\sqrt{\rho/g} v) = -\sqrt{g\rho} t + C$ . The initial condition  $v(0) = v_0$  implies that  $C = \tan^{-1}(\sqrt{\rho/g} v_0)$ , and so

$$v(t) = -\sqrt{\frac{g}{\rho}} \tan \left[ t\sqrt{g\rho} - \tan^{-1} \left( v_0 \sqrt{\frac{\rho}{g}} \right) \right].$$

We solve  $v(t) = 0$  for  $t = \frac{1}{\sqrt{g\rho}} \tan^{-1} \left( v_0 \sqrt{\frac{\rho}{g}} \right)$  and substitute in Equation (17) for  $y(t)$ :

$$\begin{aligned} y_{\max} &= \frac{1}{\rho} \ln \left| \frac{\cos \left( \tan^{-1} v_0 \sqrt{\rho/g} - \tan^{-1} v_0 \sqrt{\rho/g} \right)}{\cos \left( \tan^{-1} v_0 \sqrt{\rho/g} \right)} \right| = \frac{1}{\rho} \ln \left[ \sec \left( \tan^{-1} v_0 \sqrt{\rho/g} \right) \right] \\ &= \frac{1}{\rho} \ln \sqrt{1 + \frac{\rho v_0^2}{g}} = \frac{1}{2\rho} \ln \left( 1 + \frac{\rho v_0^2}{g} \right). \end{aligned}$$

22. By an integration similar to the one in Problem 19, the solution of the initial value problem  $v' = -32 + 0.075v^2$ ,  $v(0) = 0$  is  $v(t) = -20.666 \tanh(1.54919t)$ , so the terminal speed is 20.666 ft/sec. Then a further integration with  $y(0) = 10000$  gives

$$y(t) = 10000 - 13.333 \ln \cosh(1.54919t).$$

We solve  $y(0) = 0$  for  $t = 484.57$ . Thus the descent takes about 8 min 5 sec.

23. Before the parachute opens, the paratrooper's descent is modeled by the initial value problem  $v' = -32 + 0.00075v^2$ ,  $v(0) = 0$ , with  $y(0) = 10000$ . Solving gives  $v(t) = -206.559 \tanh(0.154919t)$ , and then  $v(30) = -206.521$  ft/sec. Integrating once again gives  $y(t) = 10000 - 1333.33 \ln(\cosh 0.154919t)$ , with  $y(30) = 4727.30$  ft. After

the parachute opens, the initial value problem becomes  $v' = -32 + 0.075v^2$ ,  $v(0) = -206.521$ , with  $y(0) = 4727.30$ . Solving gives

$$v(t) = -20.6559 \tanh(1.54919t + 0.00519595),$$

followed by

$$y(t) = 4727.30 - 13.3333 \ln(\cosh 1.54919t + 0.00519595).$$

We find that  $y = 0$  when  $t = 229.304$ . Thus he opens his parachute after 30 sec at a height of 4727 feet, and the total time of descent is  $30 + 229.304 = 259.304$  sec, about 4 minutes and 19.3 seconds.

- 24.** Let  $M$  denote the mass of the Earth. Then
- (a)  $\sqrt{2GM/R} = c$  implies  $R = 0.884 \times 10^{-3}$  meters, about 0.88 cm;
- (b)  $\sqrt{2G \cdot 329320M/R} = c$  implies  $R = 2.91 \times 10^3$  meters, about 2.91 kilometers.
- 25.** (a) The rocket's apex occurs when  $v = 0$ . We get the desired formula when we set  $v = 0$  in Eq. (23),  $v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right)$ , and solve for  $r$ .
- (b) We substitute  $v = 0$ ,  $r = R + 10^5$  (note  $100 \text{ km} = 10^5 \text{ m}$ ), and the mks values  $G = 6.6726 \times 10^{-11}$ ,  $M = 5.975 \times 10^{24}$ , and  $R = 6.378 \times 10^6$  in Eq. (23) and solve for  $v_0 = 1389.21 \text{ m/s} \approx 1.389 \text{ km/s}$ .
- (c) When we substitute  $v_0 = 0.9\sqrt{2GM/R}$  in the formula derived in part a, we find that
- $$r_{\max} = \frac{100}{19}R.$$
- 26.** By an elementary computation (as in Section 1.2) we find that an initial velocity of  $v_0 = 16$  ft/sec is required to jump vertically 4 feet high on earth. We must determine whether this initial velocity is adequate for escape from the asteroid. Let  $r$  denote the ratio of the radius of the asteroid to the radius  $R = 3960$  miles of the earth, so that

$$r = \frac{1.5}{3960} = \frac{1}{2640}.$$

Then the mass and radius of the asteroid are given by  $M_a = r^3M$  and  $R_a = rR$ , in terms of the mass  $M$  and radius  $R$  of the earth. Hence the escape velocity from the asteroid's surface is given by

$$v_a = \sqrt{\frac{2GM_a}{R_a}} = \sqrt{\frac{2G \cdot r^3M}{rR_a}} = r\sqrt{\frac{2GM}{R}} = rv_0$$

in terms of the escape velocity  $v_0$  from the earth's surface. Hence

$$v_a \approx 36680 / 2640 \approx 13.9 \text{ ft/sec}.$$

Since the escape velocity from this asteroid is thus less than the initial velocity of 16 ft/sec that your legs can provide, you can indeed jump right off this asteroid into space.

27. (a) Substitution of  $v_0^2 = \frac{2GM}{R} = \frac{k^2}{R}$  in Eq. (23) of the textbook gives

$$\frac{dr}{dt} = v = \sqrt{\frac{2GM}{r}} = \frac{k}{\sqrt{r}}.$$

We separate variables and proceed to integrate:  $\int \sqrt{r} dr = \int k dt$  implies that

$$\frac{2}{3} r^{3/2} = kt + \frac{2}{3} R^{3/2}, \text{ since } r = R \text{ when } t = 0. \text{ We solve for } r(t) = \left( \frac{2}{3} kt + R^{3/2} \right)^{2/3} \text{ and}$$

note that  $r(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

- (b) If  $v_0 > \frac{2GM}{R}$ , then Eq. (23) gives

$$\frac{dr}{dt} = v = \sqrt{\frac{2GM}{r} + \left( v_0^2 - \frac{2GM}{R} \right)} = \sqrt{\frac{k^2}{r} + \alpha} > \frac{k}{\sqrt{r}}.$$

Therefore, at every instant in its ascent, the upward velocity of the projectile in this part is greater than the velocity at the same instant of the projectile of part (a). It's as though the projectile of part (a) is the fox, and the projectile of this part is a rabbit that runs faster. Since the fox goes to infinity, so does the faster rabbit.

28. (a) We proceed as in Example 4: Since  $v = \frac{dr}{dt}$ ,  $\frac{dv}{dt}$  can be written as  $v \frac{dv}{dr}$ . Hence the given differential equation  $\frac{dv}{dt} = -\frac{GM}{r^2}$  becomes the separable equation  $v \frac{dv}{dr} = -\frac{GM}{r^2}$ .

Separating variables gives  $\int v dv = -GM \int \frac{1}{r^2} dr$ , and then integration gives

$$\frac{1}{2} v^2 = GM \left( \frac{1}{r} - \frac{1}{r_0} \right).$$

We solve for

$$\frac{dr}{dt} = v = -\sqrt{2GM \left( \frac{1}{r} - \frac{1}{r_0} \right)},$$

taking the negative square root because  $v < 0$  in descent. Again we separate variables and integrate to recover  $t$  in terms of  $r$ . Separating gives

$$\int dt = \int \frac{-1}{\sqrt{2GM \left( \frac{1}{r} - \frac{1}{r_0} \right)}} dr = -\sqrt{\frac{r_0}{2GM}} \int \sqrt{\frac{r}{r_0 - r}} dr,$$

which under the substitution  $r = r_0 \cos^2 \theta$  becomes

$$\sqrt{\frac{r_0}{2GM}} \int 2r_0 \cos^2 \theta d\theta = \frac{r_0^{3/2}}{\sqrt{2GM}} (\theta + \sin \theta \cos \theta) = \sqrt{\frac{r_0}{2GM}} \left( \sqrt{rr_0 - r^2} + r_0 \cos^{-1} \sqrt{\frac{r}{r_0}} \right).$$

(b) Substitution of  $G = 6.6726 \times 10^{-11} \text{ N} \cdot (\text{m/kg})^2$ ,  $M = 5.975 \times 10^{24} \text{ kg}$ ,

$r = R = 6.378 \times 10^6 \text{ m}$ , and  $r_0 = R + 10^6$  yields  $t = 510.504 \text{ sec}$ , that is, about  $8\frac{1}{2}$  minutes for the descent to the surface of the earth. (Recall that we are ignoring air resistance.)

(c) Substitution of the same numeral values along with  $v_0 = 0$  in the original differential equation of part (a) yields  $v = -4116.42 \text{ m/s} \approx -4.116 \text{ km/s}$  for the velocity at impact with the earth's surface, where  $r = R$ .

29. Integration of  $v \frac{dv}{dy} = -\frac{GM}{(y+R)^2}$ ,  $y(0) = 0$ ,  $v(0) = v_0$  gives

$$\frac{1}{2}v^2 = \frac{GM}{y+R} - \frac{GM}{R} + \frac{1}{2}v_0^2,$$

which simplifies to the desired formula for  $v^2$ . Then substitution of

$G = 6.6726 \times 10^{-11} \text{ N} \cdot (\text{m/kg})^2$ ,  $M = 5.975 \times 10^{24} \text{ kg}$ ,  $R = 6.378 \times 10^6 \text{ m}$ ,  $v = 0$ , and  $v_0 = 1$  yields an equation that we easily solve for  $y = 51427.3 \text{ m}$ , that is, about 51.427 km.

30. When we integrate  $v \frac{dv}{dr} = -\frac{GM_e}{r^2} + \frac{GM_m}{(S-r)^2}$ ,  $r(0) = R$ ,  $r'(0) = v_0$  in the usual way and solve for  $v$ , we get

$$v = \sqrt{\frac{2GM_e}{r} - \frac{2GM_e}{R} - \frac{2GM_m}{r-S} + \frac{2GM_m}{R-S} + v_0^2}.$$

The earth and moon attractions balance at the point where the right-hand side in the

acceleration equation vanishes, which is when  $r = \frac{\sqrt{M_e}S}{\sqrt{M_e} - \sqrt{M_m}}$ . If we substitute this

value of  $r$ ,  $M_m = 7.35 \times 10^{22} \text{ kg}$ ,  $S = 384.4 \times 10^6$ , and the usual values of the other constants involved, then set  $v = 0$  (to just reach the balancing point), we can solve the resulting equation for  $v_0 = 11,109 \text{ m/s}$ . Note that this is only 71 m/s less than the earth escape velocity of 11,180 m/s, so the moon really doesn't help much.

## SECTION 2.4

## NUMERICAL APPROXIMATION: EULER'S METHOD

In each of Problems 1–10 we also give first the explicit form of Euler's iterative formula for the given differential equation  $y' = f(x, y)$ . As we illustrate in Problem 1, the desired iterations are readily implemented, either manually or with a computer system or graphing calculator. Then we list the indicated values of  $y(\frac{1}{2})$  rounded off to 3 decimal places.

1. For the differential equation  $y' = f(x, y)$  with  $f(x, y) = -y$ , the iterative formula of Euler's method is  $y_{n+1} = y_n + h(-y_n)$ . The TI-83 screen on the left shows a graphing calculator implementation of this iterative formula.

```
0.1→H:0→X:2→Y
      2.0000
X+H→X:Y+H*(-Y)→Y
      1.8000
      1.6200
      1.4580
```

```
      1.8000
      1.6200
      1.4580
      1.3122
      1.1810
X
      .5000
```

After the variables are initialized (in the first line), and the formula is entered, each press of the enter key carries out an additional step. The screen on the right shows the results of 5 steps from  $x = 0$  to  $x = 0.5$  with step size  $h = 0.1$ —winding up with  $y(0.5) \approx 1.181$ . Similarly, using  $h = 0.25$  gives  $y(0.5) \approx 1.125$ . The true value is  $y(\frac{1}{2}) \approx 1.213$ .

The following *Mathematica* instructions produce precisely the line of data shown:

```
f[x_,y_] = -y;
g[x_] = 2*Exp[-x];
y0 = 2;
h = 0.25;
x = 0;
y1 = y0;

Do[k = f[x,y1];      (* the left-hand slope *)
  y1 = y1 + h*k;     (* Euler step to update y *)
  x = x + h,        (* update x *)
  {i,1,2}]

h = 0.1;
x = 0;
y2 = y0;

Do[k = f[x,y2];      (* the left-hand slope *)
  y2 = y2 + h*k;     (* Euler step to update y *)
```

```

    x = x + h,      (* update x *)
    {i,1,5}]

Print[x, "      ",y1,"      ",y2,"      ",g[0.5]]

0.5      1.125      1.18098      1.21306

```

2. Iterative formula:  $y_{n+1} = y_n + h(2y_n)$ ; approximate values 1.125 and 1.244; true value  $y(\frac{1}{2}) \approx 1.359$ .
3. Iterative formula:  $y_{n+1} = y_n + h(y_n + 1)$ ; approximate values 2.125 and 2.221; true value  $y(\frac{1}{2}) \approx 2.297$ .
4. Iterative formula:  $y_{n+1} = y_n + h(x_n - y_n)$ ; approximate values 0.625 and 0.681; true value  $y(\frac{1}{2}) \approx 0.713$ .
5. Iterative formula:  $y_{n+1} = y_n + h(y_n - x_n - 1)$ ; approximate values 0.938 and 0.889; true value  $y(\frac{1}{2}) \approx 0.851$ .
6. Iterative formula:  $y_{n+1} = y_n + h(-2x_n y_n)$ ; approximate values 1.750 and 1.627; true value  $y(\frac{1}{2}) \approx 1.558$ .
7. Iterative formula:  $y_{n+1} = y_n + h(-3x_n^2 y_n)$ ; approximate values 2.859 and 2.737; true value  $y(\frac{1}{2}) \approx 2.647$ .
8. Iterative formula:  $y_{n+1} = y_n + h e^{-y_n}$ ; approximate values 0.445 and 0.420; true value  $y(\frac{1}{2}) \approx 0.405$ .
9. Iterative formula:  $y_{n+1} = y_n + h \frac{1 + y_n^2}{4}$ ; approximate values 1.267 and 1.278; true value  $y(\frac{1}{2}) \approx 1.287$ .
10. Iterative formula:  $y_{n+1} = y_n + h(2x_n y_n^2)$ ; approximate values 1.125 and 1.231; true value  $y(\frac{1}{2}) \approx 1.333$ .

The tables of approximate and actual values called for in Problems 11–16 were produced using the following MATLAB script (appropriately altered for each problem).

```

% Section 2.4, Problems 11-16
x0 = 0;
y0 = 1;
% first run:
h = 0.01;
x = x0;
y = y0;
y1 = y0;
for n = 1:100
    y = y + h*(y-2);
    y1 = [y1,y];
    x = x + h;
end
% second run:
h = 0.005;
x = x0; y = y0; y2 = y0;
for n = 1:200
    y = y + h*(y-2);
    y2 = [y2,y];
    x = x + h;
end
% exact values
x = x0 : 0.2 : x0+1;
ye = 2 - exp(x);
% display table
ya = y2(1:40:201);
err = 100*(ye-ya)./ye;
[x; y1(1:20:101); ya; ye; err]

```

11. The iterative formula of Euler's method is  $y_{n+1} = y_n + h(y_n - 2)$ , and the exact solution is  $y(x) = 2 - e^x$ . The resulting table of approximate and actual values is

$x$	0.0	0.2	0.4	0.6	0.8	1.0
$y(h=0.01)$	1.0000	0.7798	0.5111	0.1833	-0.2167	-0.7048
$y(h=0.005)$	1.0000	0.7792	0.5097	0.1806	-0.2211	-0.7115
$y$ actual	1.0000	0.7786	0.5082	0.1779	-0.2255	-0.7183
error	0%	-0.08%	-0.29%	-1.53%	1.97%	0.94%

12. Iterative formula:  $y_{n+1} = y_n + h \frac{(y_n - 1)^2}{2}$ ; exact solution:  $y(x) = 1 + \frac{2}{2-x}$ .

$x$	0.0	0.2	0.4	0.6	0.8	1.0
$y(h=0.01)$	2.0000	2.1105	2.2483	2.4250	2.6597	2.9864



$y(h = 0.005)$	2.0000	2.1108	2.2491	2.4268	2.6597	2.9931
$y$ actual	2.0000	2.1111	2.2500	2.4286	2.6597	3.0000
error	0%	0.02%	0.04%	0.07%	0.13%	0.23%

13. Iterative formula:  $y_{n+1} = y_n + 2h \frac{x_n^3}{y_n}$ ; exact solution:  $y(x) = (8 + x^4)^{1/2}$ .

$x$	1.0	1.2	1.4	1.6	1.8	2.0
$y(h = 0.01)$	3.0000	3.1718	3.4368	3.8084	4.2924	4.8890
$y(h = 0.005)$	3.0000	3.1729	3.4390	3.8117	4.2967	4.8940
$y$ actual	3.0000	3.1739	3.4412	3.8149	4.3009	4.8990
error	0%	0.03%	0.06%	0.09%	0.10%	0.10%

14. Iterative formula:  $y_{n+1} = y_n + h \frac{y_n^2}{x_n}$ ; exact solution:  $y(x) = \frac{1}{1 - \ln x}$ .

$x$	1.0	1.2	1.4	1.6	1.8	2.0
$y(h = 0.01)$	1.0000	1.2215	1.5026	1.8761	2.4020	3.2031
$y(h = 0.005)$	1.0000	1.2222	1.5048	1.8814	2.4138	3.2304
$y$ actual	1.0000	1.2230	1.5071	1.8868	2.4259	3.2589
error	0%	0.06%	0.15%	0.29%	0.50%	0.87%

15. Iterative formula:  $y_{n+1} = y_n + h \left( 3 - \frac{2y_n}{x_n} \right)$ ; exact solution:  $y(x) = x + \frac{4}{x^2}$ .

$x$	2.0	2.2	2.4	2.6	2.8	3.0
$y(h = 0.01)$	3.0000	3.0253	3.0927	3.1897	3.3080	3.4422
$y(h = 0.005)$	3.0000	3.0259	3.0936	3.1907	3.3091	3.4433
$y$ actual	3.0000	3.0264	3.0944	3.1917	3.3102	3.4444
error	0%	0.019%	0.028%	0.032%	0.033%	0.032%

16. Iterative formula:  $y_{n+1} = y_n + h \left( \frac{2x_n^5}{y_n^2} \right)$ ; exact solution:  $y(x) = (x^6 - 37)^{1/3}$ .

$x$	2.0	2.2	2.4	2.6	2.8	3.0
$y(h = 0.01)$	3.0000	4.2476	5.3650	6.4805	7.6343	8.8440
$y(h = 0.005)$	3.0000	4.2452	5.3631	6.4795	7.6341	8.8445
$y$ actual	3.0000	4.2429	5.3613	6.4786	7.6340	8.8451
error	0%	-0.056%	-0.034%	-0.015%	0.002%	0.006%

The tables of approximate values called for in Problems 17–24 were produced using a MATLAB script similar to the one listed preceding the Problem 11 solution above.

17.

$x$	0.0	0.2	0.4	0.6	0.8	1.0
$y(h = 0.1)$	0.0000	0.0010	0.0140	0.0551	0.1413	0.2925
$y(h = 0.02)$	0.0000	0.0023	0.0198	0.0688	0.1672	0.3379
$y(h = 0.004)$	0.0000	0.0026	0.0210	0.0717	0.1727	0.3477
$y(h = 0.0008)$	0.0000	0.0027	0.0213	0.0723	0.1738	0.3497

These data indicate that  $y(1) \approx 0.35$ , in contrast with Example 5 in the text, where the initial condition is  $y(0) = 1$ .

In Problems 18–24 we give only the final approximate values of  $y$  obtained using Euler's method with step sizes  $h = 0.1$ ,  $h = 0.02$ ,  $h = 0.004$ , and  $h = 0.0008$ .

18. With  $x_0 = 0$  and  $y_0 = 1$ , the approximate values of  $y(2)$  obtained are:

$h$	0.1	0.02	0.004	0.0008
$y$	1.6680	1.6771	1.6790	1.6794

19. With  $x_0 = 0$  and  $y_0 = 1$ , the approximate values of  $y(2)$  obtained are:

$h$	0.1	0.02	0.004	0.0008
$y$	6.1831	6.3653	6.4022	6.4096

20. With  $x_0 = 0$  and  $y_0 = -1$ , the approximate values of  $y(2)$  obtained are:

$h$	0.1	0.02	0.004	0.0008
$y$	-1.3792	-1.2843	-1.2649	-1.2610

21. With  $x_0 = 1$  and  $y_0 = 2$ , the approximate values of  $y(2)$  obtained are:

$h$	0.1	0.02	0.004	0.0008
$y$	2.8508	2.8681	2.8716	2.8723

22. With  $x_0 = 0$  and  $y_0 = 1$ , the approximate values of  $y(2)$  obtained are:

$h$	0.1	0.02	0.004	0.0008
$y$	6.9879	7.2601	7.3154	7.3264

23. With  $x_0 = 0$  and  $y_0 = 0$ , the approximate values of  $y(1)$  obtained are:

$h$	0.1	0.02	0.004	0.0008
$y$	1.2262	1.2300	1.2306	1.2307

24. With  $x_0 = -1$  and  $y_0 = 1$ , the approximate values of  $y(1)$  obtained are:

$h$	0.1	0.02	0.004	0.0008
$y$	0.9585	0.9918	0.9984	0.9997

25. Here  $f(t, v) = 32 - 1.6v$  and  $t_0 = 0$ ,  $v_0 = 0$ . With  $h = 0.01$ , 100 iterations of  $v_{n+1} = v_n + hf(t_n, v_n)$  yield  $v(1) \approx 16.014$ , and 200 iterations with  $h = 0.005$  yield  $v(1) \approx 15.998$ . Thus we observe an approximate velocity of 16.0 ft/sec after 1 second — 80% of the limiting velocity of 20 ft/sec.

With  $h = 0.01$ , 200 iterations yield  $v(2) \approx 19.2056$ , and 400 iterations with  $h = 0.005$  yield  $v(2) \approx 19.1952$ . Thus we observe an approximate velocity of 19.2 ft/sec after 2 seconds — 96% of the limiting velocity of 20 ft/sec.

26. Here  $f(t, P) = 0.0225P - 0.003P^2$  and  $t_0 = 0$ ,  $P_0 = 25$ . With  $h = 1$ , 60 iterations of  $P_{n+1} = P_n + hf(t_n, P_n)$  yield  $P(60) \approx 49.3888$ , and 120 iterations with  $h = 0.5$  yield  $P(60) \approx 49.3903$ . Thus we observe a population of 49 deer after 5 years — 65% of the limiting population of 75 deer. With  $h = 1$ , 120 iterations yield  $P(120) \approx 66.1803$ , and 240 iterations with  $h = 0.5$  yield  $P(60) \approx 66.1469$ . Thus we observe a population of 66 deer after 10 years — 88% of the limiting population of 75 deer.

27. Here  $f(x, y) = x^2 + y^2 - 1$  and  $x_0 = 0$ ,  $y_0 = 0$ . The following table gives the approximate values for the successive step sizes  $h$  and corresponding numbers  $n$  of steps. It appears likely that  $y(2) = 1.00$  rounded off accurate to 2 decimal places.

$h$	0.1	0.01	0.001	0.0001	0.00001
$n$	20	200	2000	20000	200000
$y(2)$	0.7772	0.9777	1.0017	1.0042	1.0044

28. Here  $f(x, y) = x + \frac{1}{2}y^2$  and  $x_0 = -2$ ,  $y_0 = 0$ . The following table gives the approximate values for the successive step sizes  $h$  and corresponding numbers  $n$  of steps. It appears likely that  $y(2) = 1.46$  rounded off accurate to 2 decimal places.

$h$	0.1	0.01	0.001	0.0001	0.00001
$n$	40	400	4000	40000	400000
$y(2)$	1.2900	1.4435	1.4613	1.4631	1.4633

29. With step sizes  $h = 0.15$ ,  $h = 0.03$ , and  $h = 0.006$ , we get the following results:

$x$	$y$ with $h = 0.15$	$y$ with $h = 0.03$	$y$ with $h = 0.006$
-1.0	1.0000	1.0000	1.0000
-0.7	1.0472	1.0512	1.0521
-0.4	1.1213	1.1358	1.1390
-0.1	1.2826	1.3612	1.3835
+0.2	0.8900	1.4711	0.8210
+0.5	0.7460	1.2808	0.7192

While the values for  $h = 0.15$  alone are not conclusive, a comparison of the values of  $y$  for all three step sizes with  $x > 0$  suggests some anomaly in the transition from negative to positive values of  $x$ .

30. With step sizes  $h = 0.1$  and  $h = 0.01$  we get the following results:

$x$	$y$ with $h = 0.1$	$y$ with $h = 0.01$
0.0	0.0000	0.0000
0.1	0.0000	0.0003
0.2	0.0010	0.0025
0.3	0.0050	0.0086
$\vdots$	$\vdots$	$\vdots$
1.8	2.8200	4.3308
1.9	3.9393	7.9425
2.0	5.8521	28.3926

Clearly there is some difficulty near  $x = 2$ .

31. With step sizes  $h = 0.1$  and  $h = 0.01$  we get the following results:

$x$	$y$ with $h = 0.1$	$y$ with $h = 0.01$
0.0	1.0000	1.0000
0.1	1.2000	1.2200
0.2	1.4428	1.4967
$\vdots$	$\vdots$	$\vdots$
0.7	4.3460	6.4643
0.8	5.8670	11.8425
0.9	8.3349	39.5010

Clearly there is some difficulty near  $x = 0.9$ .

## SECTION 2.5

## A CLOSER LOOK AT THE EULER METHOD

In each of Problems 1–10 we give first the predictor formula for  $u_{n+1}$  and then the improved Euler corrector for  $y_{n+1}$ . These predictor-corrector iterations are readily implemented, either manually or with a computer system or graphing calculator (as we illustrate in Problem 1). We give in each problem a table showing the approximate values obtained, as well as the corresponding values of the exact solution.

```

Y-H*Y→U:Y+(H/2)*
(-Y-U)→Y
1.8100
1.6381
1.4824
1.3416
1.2142

```

```

0.1→H:0→X:2→Y
2.0000
Y-H*Y→U:Y+(H/2)*
(-Y-U)→Y
1.8100
1.6381
1.4824

```

$$1. \quad u_{n+1} = y_n + h(-y_n); \quad y_{n+1} = y_n + \frac{h}{2}(-y_n - u_{n+1})$$

The TI-83 screen on the left above shows a graphing calculator implementation of this iteration. After the variables are initialized (in the first line), and the formulas are entered, each press of the enter key carries out an additional step. The screen on the right shows the results of 5 steps from  $x = 0$  to  $x = 0.5$  with step size  $h = 0.1$  — winding up with  $y(0.5) \approx 1.2142$  — and we see the approximate values shown in the second row of the table below.

$x$	0.0	0.1	0.2	0.3	0.4	0.5
$y$ with $h = 0.1$	2.0000	1.8100	1.6381	1.4824	1.3416	1.2142
$y$ actual	2.0000	1.8097	1.6375	1.4816	1.3406	1.2131

$$2. \quad u_{n+1} = y_n + 2hy_n; \quad y_{n+1} = y_n + \frac{h}{2}(2y_n + 2u_{n+1})$$

$x$	0.0	0.1	0.2	0.3	0.4	0.5
$y$ with $h = 0.1$	0.5000	0.6100	0.7422	0.9079	1.1077	1.3514
$y$ actual	0.5000	0.6107	0.7459	0.9111	1.1128	1.3591

$$3. \quad u_{n+1} = y_n + h(y_n + 1); \quad y_{n+1} = y_n + \frac{h}{2}[(y_n + 1) + (u_{n+1} + 1)]$$

$x$	0.0	0.1	0.2	0.3	0.4	0.5
$y$ with $h = 0.1$	1.0000	1.2100	1.4421	1.6985	1.9818	2.2949
$y$ actual	1.0000	1.2103	1.4428	1.6997	1.9837	2.2974

$$4. \quad u_{n+1} = y_n + h(x_n - y_n); \quad y_{n+1} = y_n + \frac{h}{2}[(x_n - y_n) + (x_n + h - u_{n+1})]$$

$x$	0.0	0.1	0.2	0.3	0.4	0.5
$y$ with $h = 0.1$	1.0000	0.9100	0.8381	0.7824	0.7416	0.7142
$y$ actual	1.0000	0.9097	0.8375	0.7816	0.7406	0.7131

$$5. \quad u_{n+1} = y_n + h(y_n - x_n - 1); \quad y_{n+1} = y_n + \frac{h}{2}[(y_n - x_n - 1) + (u_{n+1} - x_n - h - 1)]$$

$x$	0.0	0.1	0.2	0.3	0.4	0.5
$y$ with $h = 0.1$	1.0000	0.9950	0.9790	0.9508	0.9091	0.8526
$y$ actual	1.0000	0.9948	0.9786	0.9501	0.9082	0.8513

$$6. \quad u_{n+1} = y_n - 2x_n y_n h; \quad y_{n+1} = y_n + \frac{h}{2}[2x_n y_n + 2(x_n + h)u_{n+1}]$$

$x$	0.0	0.1	0.2	0.3	0.4	0.5
$y$ with $h = 0.1$	2.0000	1.9800	1.9214	1.8276	1.7041	1.5575
$y$ actual	2.0000	1.9801	1.9216	1.8279	1.7043	1.5576

$$7. \quad u_{n+1} = y_n - 3x_n^2 y_n h; \quad y_{n+1} = y_n - \frac{h}{2}[3x_n^2 y_n + 3(x_n + h)^2 u_{n+1}]$$

$x$	0.0	0.1	0.2	0.3	0.4	0.5
$y$ with $h = 0.1$	3.0000	2.9955	2.9731	2.9156	2.8082	2.6405
$y$ actual	3.0000	2.9970	2.9761	2.9201	2.8140	2.6475

$$8. \quad u_{n+1} = y_n + h e^{-y_n}; \quad y_{n+1} = y_n + \frac{h}{2}[e^{-y_n} + e^{-u_{n+1}}]$$

$x$	0.0	0.1	0.2	0.3	0.4	0.5
$y$ with $h = 0.1$	0.0000	0.0952	0.1822	0.2622	0.3363	0.4053
$y$ actual	0.0000	0.0953	0.1823	0.2624	0.3365	0.4055

$$9. \quad u_{n+1} = y_n + h \cdot \frac{1+y_n^2}{4}; \quad y_{n+1} = y_n + h \cdot \frac{1+y_n^2+1+u_{n+1}^2}{8}$$

$x$	0.0	0.1	0.2	0.3	0.4	0.5
$y$ with $h = 0.1$	1.0000	1.0513	1.1053	1.1625	1.2230	1.2873
$y$ actual	1.0000	1.0513	1.1054	1.1625	1.2231	1.2874

$$10. \quad u_{n+1} = y_n + h \cdot 2x_n y_n^2; \quad y_{n+1} = y_n + h \cdot [x_n y_n^2 + (x_n + h) u_{n+1}^2]$$

$x$	0.0	0.1	0.2	0.3	0.4	0.5
$y$ with $h = 0.1$	1.0000	1.0100	1.0414	1.0984	1.1895	1.3309
$y$ actual	1.0000	1.0101	1.0417	1.0989	1.1905	1.3333

The results given below for Problems 11–16 were computed using the following MATLAB script.

```

% Section 2.5, Problems 11-16
x0 = 0;  y0 = 1;
% first run:
h = 0.01;
x = x0;  y = y0;  y1 = y0;
for n = 1:100
    u = y + h*f(x,y);                                     %predictor
    y = y + (h/2)*(f(x,y)+f(x+h,u));                     %corrector
    y1 = [y1,y];
    x = x + h;
end
% second run:
h = 0.005;
x = x0;  y = y0;  y2 = y0;

for n = 1:200
    u = y + h*f(x,y);                                     %predictor
    y = y + (h/2)*(f(x,y)+f(x+h,u));                     %corrector
    y2 = [y2,y];
    x = x + h;
end

% exact values
x = x0 : 0.2 : x0+1;
ye = g(x);

% display table
ya = y2(1:40:201);

```

```

err = 100*(ye-ya)./ye;
x = sprintf('%10.5f',x), sprintf('\n');
y1 = sprintf('%10.5f',y1(1:20:101)), sprintf('\n');
ya = sprintf('%10.5f',ya), sprintf('\n');
ye = sprintf('%10.5f',ye), sprintf('\n');
err = sprintf('%10.5f',err), sprintf('\n');
table = [x; y1; ya; ye; err]

```

For each problem the differential equation  $y' = f(x, y)$  and the known exact solution  $y = g(x)$  are stored in the files **f.m** and **g.m** — for instance, the files

```

function yp = f(x,y)
yp = y-2;

function ye = g(x,y)
ye = 2-exp(x);

```

for Problem 11. (The exact solutions for Problems 11–16 here are given in the solutions for Problems 11–16 in Section 2.4.)

**11.**

$x$	0.0	0.2	0.4	0.6	0.8	1.0
$y(h = 0.01)$	1.00000	0.77860	0.50819	0.17790	-0.22551	-0.71824
$y(h = 0.005)$	1.00000	0.77860	0.50818	0.17789	-0.22553	-0.71827
$y$ actual	1.00000	0.77860	0.50818	0.17788	-0.22554	-0.71828
error	0.000%	-0.000%	-0.000%	-0.003%	0.003%	0.002%

**12.**

$x$	0.0	0.2	0.4	0.6	0.8	1.0
$y(h = 0.01)$	2.00000	2.11111	2.25000	2.42856	2.66664	2.99995
$y(h = 0.005)$	2.00000	2.11111	2.25000	2.42857	2.66666	2.99999
$y$ actual	2.00000	2.11111	2.25000	2.42857	2.66667	3.00000
error	0.0000%	0.0000%	0.0001%	0.0001%	0.0002%	0.0004%

**13.**

$x$	1.0	1.2	1.4	1.6	1.8	2.0
$y(h = 0.01)$	3.00000	3.17390	3.44118	3.81494	4.30091	4.89901
$y(h = 0.005)$	3.00000	3.17390	3.44117	3.81492	4.30089	4.89899
$y$ actual	3.00000	3.17389	3.44116	3.81492	4.30088	4.89898
error	0.0000%	-0.0001%	-0.0001%	-0.0001%	-0.0002%	-0.0002%



14.

$x$	1.0	1.2	1.4	1.6	1.8	2.0
$y(h = 0.01)$	1.00000	1.22296	1.50707	1.88673	2.42576	3.25847
$y(h = 0.005)$	1.00000	1.22297	1.50709	1.88679	2.42589	3.25878
$y$ actual	1.00000	1.22297	1.50710	1.88681	2.42593	3.25889
error	0.00000%	0.0002%	0.0005%	0.0010%	0.0018%	0.0033%

15.

$x$	2.0	2.2	2.4	2.6	2.8	3.0
$y(h = 0.01)$	3.000000	3.026448	3.094447	3.191719	3.310207	3.444448
$y(h = 0.005)$	3.000000	3.026447	3.094445	3.191717	3.310205	3.444445
$y$ actual	3.000000	3.026446	3.094444	3.191716	3.310204	3.444444
error	0.00000%	-0.00002%	-0.00002%	-0.00002%	-0.00002%	-0.00002%

16.

$x$	2.0	2.2	2.4	2.6	2.8	3.0
$y(h = 0.01)$	3.000000	4.242859	5.361304	6.478567	7.633999	8.845112
$y(h = 0.005)$	3.000000	4.242867	5.361303	6.478558	7.633984	8.845092
$y$ actual	3.000000	4.242870	5.361303	6.478555	7.633979	8.845085
error	0.00000%	0.00006%	-0.00001%	-0.00005%	-0.00007%	-0.00007%

17.

With $h = 0.1$ :	$y(1) \approx 0.35183$
With $h = 0.02$ :	$y(1) \approx 0.35030$
With $h = 0.004$ :	$y(1) \approx 0.35023$
With $h = 0.0008$ :	$y(1) \approx 0.35023$

The table of numerical results is

$x$	$y$ with $h = 0.1$	$y$ with $h = 0.02$	$y$ with $h = 0.004$	$y$ with $h = 0.0008$
0.0	0.00000	0.00000	0.00000	0.00000
0.2	0.00300	0.00268	0.00267	0.00267
0.4	0.02202	0.02139	0.02136	0.02136
0.6	0.07344	0.07249	0.07245	0.07245
0.8	0.17540	0.17413	0.17408	0.17408
1.0	0.35183	0.35030	0.35023	0.35023

In Problems 18–24 we give only the final approximate values of  $y$  obtained using the improved Euler method with step sizes  $h = 0.1$ ,  $h = 0.02$ ,  $h = 0.004$ , and  $h = 0.0008$ .

18.

Value of $h$	Estimated value of $y(2)$
0.1	1.68043
0.02	1.67949
0.004	1.67946
0.0008	1.67946

19.

Value of $h$	Estimated value of $y(2)$
0.1	6.40834
0.02	6.41134
0.004	6.41147
0.0008	6.41147

20.

Value of $h$	Estimated value of $y(2)$
0.1	-1.26092
0.02	-1.26003
0.004	-1.25999
0.0008	-1.25999

21.

Value of $h$	Estimated value of $y(2)$
0.1	2.87204
0.02	2.87245
0.004	2.87247
0.0008	2.87247

22.

Value of $h$	Estimated value of $y(2)$
0.1	7.31578
0.02	7.32841
0.004	7.32916
0.0008	7.32920

23.

Value of $h$	Estimated value of $y(1)$
0.1	1.22967
0.02	1.23069
0.004	1.23073
0.0008	1.23073

24.

Value of $h$	Estimated value of $y(1)$
0.1	1.00006
0.02	1.00000
0.004	1.00000
0.0008	1.00000

25. Here  $f(t, v) = 32 - 1.6v$  and  $t_0 = 0$ ,  $v_0 = 0$ . With  $h = 0.01$ , 100 iterations of

$$k_1 = f(t, v_n), k_2 = f(t + h, v_n + hk_1), v_{n+1} = v_n + \frac{h}{2}(k_1 + k_2)$$

yield  $v(1) \approx 15.9618$ , and 200 iterations with  $h = 0.005$  yield  $v(1) \approx 15.9620$ . Thus we observe an approximate velocity of 15.962 ft/sec after 1 second — 80% of the limiting velocity of 20 ft/sec.

With  $h = 0.01$ , 200 iterations yield  $v(2) \approx 19.1846$ , and 400 iterations with  $h = 0.005$  yield  $v(2) \approx 19.1847$ . Thus we observe an approximate velocity of 19.185 ft/sec after 2 seconds — 96% of the limiting velocity of 20 ft/sec.

26. Here  $f(t, P) = 0.0225P - 0.003P^2$  and  $t_0 = 0$ ,  $P_0 = 25$ . With  $h = 1$ , 60 iterations of

$$k_1 = f(t, P_n), k_2 = f(t + h, P_n + hk_1), P_{n+1} = P_n + \frac{h}{2}(k_1 + k_2)$$

yield  $P(60) \approx 49.3909$ , and 120 iterations with  $h = 0.5$  yield  $P(60) \approx 49.3913$ . Thus we observe an approximate population of 49.391 deer after 5 years — 65% of the limiting population of 75 deer.

With  $h = 1$ , 120 iterations yield  $P(120) \approx 66.1129$ , and 240 iterations with  $h = 0.5$  yield  $P(120) \approx 66.1134$ . Thus we observe an approximate population of 66.113 deer after 10 years — 88% of the limiting population of 75 deer.

27. Here  $f(x, y) = x^2 + y^2 - 1$  and  $x_0 = 0$ ,  $y_0 = 0$ . The following table gives the approximate values for the successive step sizes  $h$  and corresponding numbers  $n$  of steps. It appears likely that  $y(2) = 1.0045$  rounded off accurate to 4 decimal places.

$h$	0.1	0.01	0.001	0.0001
$n$	20	200	2000	20000
$y(2)$	1.01087	1.00452	1.00445	1.00445

28. Here  $f(x, y) = x + \frac{1}{2}y^2$  and  $x_0 = -2$ ,  $y_0 = 0$ . The following table gives the approximate values for the successive step sizes  $h$  and corresponding numbers  $n$  of steps. It appears likely that  $y(2) = 1.4633$  rounded off accurate to 4 decimal places.

$h$	0.1	0.01	0.001	0.0001
$n$	40	400	4000	40000
$y(2)$	1.46620	1.46335	1.46332	1.46331

In the solutions for Problems 29 and 30 we illustrate the following general MATLAB ode solver.

```
function [t,y] = ode(method, yp, t0,b, y0, n)
% [t,y] = ode(method, yp, t0,b, y0, n)
% calls the method described by 'method' for the
% ODE 'yp' with function header
%
%           y' = yp(t,y)
%
% on the interval [t0,b] with initial (column)
% vector y0. Choices for method are 'euler',
% 'impeuler', 'rk' (Runge-Kutta), 'ode23', 'ode45'.
% Results are saved at the endPoints of n subintervals,
% that is, in steps of length h = (b - t0)/n. The
% result t is an (n+1)-column vector from b to t1,
% while y is a matrix with n+1 rows (one for each
% t-value) and one column for each dependent variable.

h = (b - t0)/n;           % step size
t = t0 : h : b;
t = t';                   % col. vector of t-values
y = y0';                  % 1st row of result matrix
```

```

for i = 2 : n+1           % for i=2 to i=n+1
    t0 = t(i-1);         % old t
    t1 = t(i);           % new t
    y0 = y(i-1,:);       % old y-row-vector
    [T,Y] = feval(method, yp, t0,t1, y0);
    y = [y;Y'];          % adjoin new y-row-vector
end

```

To use the improved Euler method, we call as '**method**' the following function.

```

function [t,y] = impeuler(yp, t0,t1, y0)
%
% [t,y] = impeuler(yp, t0,t1, y0)
% Takes one improved Euler step for
%
%     y' = yprime( t,y ),
%
% from t0 to t1 with initial value the
% column vector y0.

h = t1 - t0;
k1 = feval( yp, t0, y0      );
k2 = feval( yp, t1, y0 + h*k1 );
k  = (k1 + k2)/2;
t  = t1;
y  = y0 + h*k;

```

29. Here our differential equation is described by the MATLAB function

```

function vp = vpbolt1(t,v)
vp = -0.04*v - 9.8;

```

Then the commands

```

n = 50;
[t1,v1] = ode('impeuler','vpbolt1',0,10,49,n);
n = 100;
[t2,v2] = ode('impeuler','vpbolt1',0,10,49,n);
t = (0:10)';
ve = 294*exp(-t/25)-245;
[t, v1(1:5:51), v2(1:10:101), ve]

```

generate the table:

$t$	with $n = 50$	with $n = 100$	actual $v$
0	49.0000	49.0000	49.0000
1	37.4722	37.4721	37.4721
2	26.3964	26.3963	26.3962
3	15.7549	15.7547	15.7546
4	5.5307	5.5304	5.5303
5	-4.2926	-4.2930	-4.2932
6	-13.7308	-13.7313	-13.7314
7	-22.7989	-22.7994	-22.7996
8	-31.5115	-31.5120	-31.5122
9	-39.8824	-39.8830	-39.8832
10	-47.9251	-47.9257	-47.9259

We notice first that the final two columns agree to 3 decimal places (each difference being less than 0.0005). Scanning the  $n = 100$  column for sign changes, we suspect that  $v = 100$  (at the bolt's apex) occurs just after  $t = 4.5$  sec. Then interpolation between  $t = 4.5$  and  $t = 4.6$  in the table

`[t2(40:51), v2(40:51)]`

3.9000	6.5345
4.0000	5.5304
4.1000	4.5303
4.2000	3.5341
4.3000	2.5420
4.4000	1.5538
4.5000	0.5696
4.6000	-0.4108
4.7000	-1.3872
4.8000	-2.3597
4.9000	-3.3283
5.0000	-4.2930

indicates that  $t = 4.56$  at the bolt's apex. Finally, interpolation in

`[t2(95:96), v2(95:96)]`

9.4000	-43.1387
9.5000	-43.9445

gives the impact velocity  $v(9.41) \approx -43.22$  m/s.

30. Now our differential equation is described by the MATLAB function

```
function vp = vpbolt2(t,v)
vp = -0.0011*v.*abs(v) - 9.8;
```

Then the commands

```

n = 100;
[t1,v1] = ode('impeuler','vpbolt2',0,10,49,n);
n = 200;
[t2,v2] = ode('impeuler','vpbolt2',0,10,49,n);
t = (0:10)';
[t, v1(1:10:101), v2(1:20:201)]

```

generate the table

$t$	with $n = 100$	with $n = 200$
0	49.0000	49.0000
1	37.1547	37.1547
2	26.2428	26.2429
3	15.9453	15.9455
4	6.0041	6.0044
5	-3.8020	-3.8016
6	-13.5105	-13.5102
7	-22.9356	-22.9355
8	-31.8984	-31.8985
9	-40.2557	-40.2559
10	-47.9066	-47.9070

We notice first that the final two columns agree to 2 decimal places (each difference being less than 0.005). Scanning the  $n = 200$  column for sign changes, we suspect that  $v = 0$  (at the bolt's apex) occurs just after  $t = 4.6$  sec. Then interpolation between  $t = 4.60$  and  $t = 4.65$   $t = 4.60$  in the table

```
[t2(91:101),v2(91:101)]
```

4.5000	1.0964
4.5500	0.6063
4.6000	0.1163
4.6500	-0.3737
4.7000	-0.8636
4.7500	-1.3536
4.8000	-1.8434
4.8500	-2.3332
4.9000	-2.8228
4.9500	-3.3123
5.0000	-3.8016

indicates that  $T = 4.61$  at the bolt's apex. Finally, interpolation in 

```
[t2(189:190),v2(189:190)]
```

9.4000	-43.4052
9.4500	-43.7907

gives the impact velocity  $v(9.41) \approx -43.48 \text{ m/s}$ .

## SECTION 2.6

### THE RUNGE-KUTTA METHOD

Each problem can be solved with a “template” of computations like those listed in Problem 1. We include a table showing the slope values  $k_1, k_2, k_3, k_4$  and the  $xy$ -values at the ends of two successive steps of size  $h = 0.25$ .

1. To make the first step of size  $h = 0.25$  we start with the function defined by

`f[x_, y_] := -y`

and the initial values

`x = 0; y = 2; h = 0.25;`

and then perform the calculations

`k1 = f[x, y]`

`k2 = f[x + h/2, y + h*k1/2]`

`k3 = f[x + h/2, y + h*k2/2]`

`k4 = f[x + h, y + h*k3]`

`y = y + (h/6)*(k1 + 2*k2 + 2*k3 + k4)`

`x = x + h`

in turn. Here we are using *Mathematica* notation that translates transparently to standard mathematical notation describing the corresponding manual computations. A repetition of this same block of calculations carries out a second step of size  $h = 0.25$ . The following table lists the intermediate and final results obtained in these two steps.

$k_1$	$k_2$	$k_3$	$k_4$	$x$	Approx. $y$	Actual $y$
-2	-1/75	-1.78125	-1.55469	0.25	1.55762	1.55760
-1.55762	-1.36292	-1.38725	-1.2108	0.5	1.21309	1.21306

- 2.

$k_1$	$k_2$	$k_3$	$k_4$	$x$	Approx. $y$	Actual $y$
1	1.25	1.3125	1.65625	0.25	0.82422	0.82436
1.64844	2.06055	2.16357	2.73022	0.5	1.35867	1.35914

- 3.

$k_1$	$k_2$	$k_3$	$k_4$	$x$	Approx. $y$	Actual $y$
2	2.25	2.28125	2.57031	0.25	1.56803	1.56805
2.56803	2.88904	2.92916	3.30032	0.5	2.29740	2.29744



4.

$k_1$	$k_2$	$k_3$	$k_4$	$x$	Approx. $y$	Actual $y$
-1	-0.75	-0.78128	-55469	0.25	0.80762	0.80760
-0.55762	-0.36292	-0.38725	-0.21080	0.5	0.71309	0.71306

5.

$k_1$	$k_2$	$k_3$	$k_4$	$x$	Approx. $y$	Actual $y$
0	-0.125	-0.14063	-0.28516	0.25	0.96598	0.96597
-28402	-0.44452	-0.46458	-0.65016	0.5	0.85130	0.85128

6.

$k_1$	$k_2$	$k_3$	$k_4$	$x$	Approx. $y$	Actual $y$
0	-0.5	-0.48438	-0.93945	0.25	1.87882	1.87883
-0.93941	-1.32105	-1.28527	-1.55751	0.5	1.55759	1.55760

7.

$k_1$	$k_2$	$k_3$	$k_4$	$x$	Approx. $y$	Actual $y$
0	-0.14063	-0.13980	-0.55595	0.25	2.95347	2.95349
-0.55378	-1.21679	-1.18183	-1.99351	0.5	2.6475	2.64749

8.

$k_1$	$k_2$	$k_3$	$k_4$	$x$	Approx. $y$	Actual $y$
1	0.88250	0.89556	0.79940	0.25	0.22315	0.22314
0.80000	0.72387	0.73079	0.66641	0.5	0.40547	0.40547

9.

$k_1$	$k_2$	$k_3$	$k_4$	$x$	Approx. $y$	Actual $y$
0.5	0.53223	0.53437	0.57126	0.25	1.13352	1.13352
0.57122	0.61296	0.61611	0.66444	0.5	1.28743	1.28743

10.

$k_1$	$k_2$	$k_3$	$k_4$	$x$	Approx. $y$	Actual $y$
0	0.25	0.26587	0.56868	0.25	1.06668	1.06667
0.56891	0.97094	1.05860	1.77245	0.5	1.33337	1.33333

The results given below for Problems 11–16 were computed using the following MATLAB script.

```

% Section 2.6, Problems 11-16
x0 = 0;  y0 = 1;

% first run:
h = 0.2;
x = x0;  y = y0;  y1 = y0;
for n = 1:5
    k1 = f(x,y);
    k2 = f(x+h/2,y+h*k1/2);
    k3 = f(x+h/2,y+h*k2/2);
    k4 = f(x+h,y+h*k3);
    y = y + (h/6)*(k1+2*k2+2*k3+k4);
    y1 = [y1,y];
    x = x + h;
end

% second run:
h = 0.1;
x = x0;  y = y0;  y2 = y0;
for n = 1:10
    k1 = f(x,y);
    k2 = f(x+h/2,y+h*k1/2);
    k3 = f(x+h/2,y+h*k2/2);
    k4 = f(x+h,y+h*k3);
    y = y + (h/6)*(k1+2*k2+2*k3+k4);
    y2 = [y2,y];
    x = x + h;
end

% exact values
x = x0 : 0.2 : x0+1;
ye = g(x);

% display table
y2 = y2(1:2:11);
err = 100*(ye-y2)./ye;
x = sprintf('%10.6f',x), sprintf('\n');
y1 = sprintf('%10.6f',y1), sprintf('\n');
y2 = sprintf('%10.6f',y2), sprintf('\n');
ye = sprintf('%10.6f',ye), sprintf('\n');
err = sprintf('%10.6f',err), sprintf('\n');
table = [x;y1;y2;ye;err]

```

For each problem the differential equation  $y' = f(x, y)$  and the known exact solution  $y = g(x)$  are stored in the files **f.m** and **g.m** — for instance, the files

```
function yp = f(x,y)
yp = y-2;
```

and

```
function ye = g(x,y)
ye = 2-exp(x);
```

for Problem 11.

11.

$x$	0.0	0.2	0.4	0.6	0.8	1.0
$y(h = 0.2)$	1.000000	0.778600	0.508182	0.177894	-0.225521	-0.718251
$y(h = 0.1)$	1.000000	0.778597	0.508176	0.177882	-0.225540	-0.718280
$y$ actual	1.000000	0.778597	0.508175	0.177881	-0.225541	-0.718282
error	0.000000%	-0.00002%	-0.00009%	-0.00047%	-0.00061%	-0.00029%

12.

$x$	0.0	0.2	0.4	0.6	0.8	1.0
$y(h = 0.2)$	2.000000	2.111110	2.249998	2.428566	2.666653	2.999963
$y(h = 0.1)$	2.000000	2.111111	2.250000	2.428571	2.666666	2.999998
$y$ actual	2.000000	2.111111	2.250000	2.428571	2.666667	3.000000
error	0.000000%	0.000002%	0.000006%	0.000014%	0.000032%	0.000080%

13.

$x$	1.0	1.2	1.4	1.6	1.8	2.0
$y(h = 0.2)$	3.000000	3.173896	3.441170	3.814932	4.300904	4.899004
$y(h = 0.1)$	3.000000	3.173894	3.441163	3.814919	4.300885	4.898981
$y$ actual	3.000000	3.173894	3.441163	3.814918	4.300884	4.898979
error	0.000000%	-0.00001%	-0.00001%	-0.00002%	-0.00003%	-0.00003%

14.

$x$	1.0	1.2	1.4	1.6	1.8	2.0
$y(h = 0.2)$	1.000000	1.222957	1.507040	1.886667	2.425586	3.257946
$y(h = 0.1)$	1.000000	1.222973	1.507092	1.886795	2.425903	3.258821
$y$ actual	1.000000	1.222975	1.507096	1.886805	2.425928	3.258891
error	0.0000%	0.0001%	0.0003%	0.0005%	0.0010%	-0.0021%

15.

$x$	2.0	2.2	2.4	2.6	2.9	3.0
$y(h = 0.2)$	3.000000	3.026448	3.094447	3.191719	3.310207	3.444447
$y(h = 0.1)$	3.000000	3.026446	3.094445	3.191716	3.310204	3.444445

$y$ actual	3.000000	3.026446	3.094444	3.191716	3.310204	3.444444
error	-0.000000%	-0.000004%	-0.000005%	-0.000005%	-0.000005%	-0.000004%

16.

$x$	2.0	2.2	2.4	2.6	2.9	3.0
$y (h = 0.2)$	3.000000	4.243067	5.361409	6.478634	7.634049	8.845150
$y (h = 0.1)$	3.000000	4.242879	5.361308	6.478559	7.633983	8.845089
$y$ actual	3.000000	4.242870	5.361303	6.478555	7.633979	8.845085
error	-0.000000%	-0.000221%	-0.000094%	-0.000061%	-0.000047%	-0.000039%

17.

Value of $h$	Estimated value of $y(1)$
0.2	0.350258
0.1	0.350234
0.05	0.350232
0.025	0.350232

The table of numerical results is

$x$	$y$ with $h = 0.2$	$y$ with $h = 0.1$	$y$ with $h = 0.05$	$y$ with $h = 0.025$
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.002667	0.002667	0.002667	0.002667
0.4	0.021360	0.021359	0.021359	0.021359
0.6	0.072451	0.072448	0.072448	0.072448
0.8	0.174090	0.174081	0.174080	0.174080
1.0	0.350258	0.350234	0.350232	0.350232

In Problems 18–24 we give only the final approximate values of  $y$  obtained using the Runge-Kutta method with step sizes  $h = 0.2$ ,  $h = 0.1$ ,  $h = 0.05$ , and  $h = 0.025$ .

18.

Value of $h$	Estimated value of $y(2)$
0.2	1.679513
0.1	1.679461
0.05	1.679459
0.025	1.679459

19.

Value of $h$	Estimated value of $y(2)$
0.2	1.679459
0.1	6.411474
0.05	6.411474
0.025	6.411474

20.

Value of $h$	Estimated value of $y(2)$
0.2	-1.259990
0.1	-1.259992
0.05	-1.259993
0.025	-1.259993

21.

Value of $h$	Estimated value of $y(2)$
0.2	2.872467
0.1	2.872468
0.05	2.872468
0.025	2.872468

22.

Value of $h$	Estimated value of $y(2)$
0.2	7.326761
0.1	7.328452
0.05	7.328971
0.025	7.329134

23.

Value of $h$	Estimated value of $y(1)$
0.2	1.230735
0.1	1.230731
0.05	1.230731
0.025	1.230731

24.

Value of $h$	Estimated value of $y(1)$
0.2	1.000000
0.1	1.000000
0.05	1.000000
0.025	1.000000

25. Here  $f(t, v) = 32 - 1.6v$  and  $t_0 = 0$ ,  $v_0 = 0$ . With  $h = 0.1$ , 10 iterations of

$$\begin{aligned}
 k_1 &= f(t_n, v_n) & k_2 &= f\left(t_n + \frac{1}{2}h, v_n + \frac{1}{2}hk_1\right) \\
 k_3 &= f\left(t_n + \frac{1}{2}h, v_n + \frac{1}{2}hk_2\right) & k_4 &= f(t_n + h, v_n + hk_3) \\
 k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) & v_{n+1} &= v_n + hk
 \end{aligned}$$

yield  $v(1) \approx 15.9620$ , and 20 iterations with  $h = 0.05$  yield  $v(1) \approx 15.9621$ . Thus we observe an approximate velocity of 15.962 ft/sec after 1 second — 80% of the limiting velocity of 20 ft/sec.

With  $h = 0.1$ , 20 iterations yield  $v(2) \approx 19.1847$ , and 40 iterations with  $h = 0.05$  yield  $v(2) \approx 19.1848$ . Thus we observe an approximate velocity of 19.185 ft/sec after 2 seconds — 96% of the limiting velocity of 20 ft/sec.

26. Here  $f(t, P) = 0.0225P - 0.003P^2$  and  $t_0 = 0$ ,  $P_0 = 25$ . With  $h = 6$ , 10 iterations of

$$\begin{aligned}
 k_1 &= f(t_n, P_n) & k_2 &= f\left(t_n + \frac{1}{2}h, P_n + \frac{1}{2}hk_1\right) \\
 k_3 &= f\left(t_n + \frac{1}{2}h, P_n + \frac{1}{2}hk_2\right) & k_4 &= f(t_n + h, P_n + hk_3) \\
 k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) & P_{n+1} &= P_n + hk
 \end{aligned}$$

yield  $P(60) \approx 49.3915$ , as do 20 iterations with  $h = 3$ . Thus we observe an approximate population of 49.3915 deer after 5 years — 65% of the limiting population of 75 deer.

With  $h = 6$ , 20 iterations yield  $P(120) \approx 66.1136$ , as do 40 iterations with  $h = 3$ . Thus we observe an approximate population of 66.1136 deer after 10 years — 88% of the limiting population of 75 deer.

27. Here  $f(x, y) = x^2 + y^2 - 1$  and  $x_0 = 0$ ,  $y_0 = 0$ . The following table gives the approximate values for the successive step sizes  $h$  and corresponding numbers  $n$  of steps. It appears likely that  $y(2) = 1.00445$  rounded off accurate to 5 decimal places.

$h$	1	0.1	0.01	0.001
$n$	2	20	200	2000
$y(2)$	1.05722	1.00447	1.00445	1.00445

28. Here  $f(x, y) = x + \frac{1}{2}y^2$  and  $x_0 = -2$ ,  $y_0 = 0$ . The following table gives the approximate values for the successive step sizes  $h$  and corresponding numbers  $n$  of steps. It appears likely that  $y(2) = 1.46331$  rounded off accurate to 5 decimal places.

$h$	1	0.1	0.01	0.001
$n$	4	40	400	40000
$y(2)$	1.48990	1.46332	1.46331	1.46331

In the solutions for Problems 29 and 30 we use the general MATLAB solver **ode** that was listed prior to the Problem 29 solution in Section 2.5. To use the Runge-Kutta method, we call as '**method**' the following function.

```
function [t,y] = rk(yp, t0,t1, y0)

% [t, y] = rk(yp, t0, t1, y0)
% Takes one Runge-Kutta step for
%
%      y' = yp( t,y ),
%
% from t0 to t1 with initial value the
% column vector y0.

h = t1 - t0;
k1 = feval(yp, t0, y0);
k2 = feval(yp, t0 + h/2, y0 + (h/2)*k1);
k3 = feval(yp, t0 + h/2, y0 + (h/2)*k2);
k4 = feval(yp, t0 + h, y0 + h*k3);
k = (1/6)*(k1 + 2*k2 + 2*k3 + k4);
t = t1;
y = y0 + h*k;
```

29. Here our differential equation is described by the MATLAB function

```
function vp = vpbolt1(t,v)
vp = -0.04*v - 9.8;
```

Then the commands

```
n = 100;
[t1,v1] = ode('rk', 'vpbolt1', 0, 10, 49, n);
n = 200;
```

```
[t2,v] = ode('rk','vpbolt1',0,10,49,n);
t = (0:10)';
ve = 294*exp(-t/25)-245;
[t, v1(1:n/20:1+n/2), v(1:n/10:n+1), ve]
```

generate the table

$t$	with $n = 100$	with $n = 200$	actual $v$
0	49.0000	49.0000	49.0000
1	37.4721	37.4721	37.4721
2	26.3962	26.3962	26.3962
3	15.7546	15.7546	15.7546
4	5.5303	5.5303	5.5303
5	-4.2932	-4.2932	-4.2932
6	-13.7314	-13.7314	-13.7314
7	-22.7996	-22.7996	-22.7996
8	-31.5122	-31.5122	-31.5122
9	-39.8832	-39.8832	-39.8832
10	-47.9259	-47.9259	-47.9259

We notice first that the final three columns agree to the 4 displayed decimal places. Scanning the last column for sign changes in  $v$ , we suspect that  $v = 0$  (at the bolt's apex) occurs just after  $t = 4.5$ sec. Then interpolation between  $t = 4.55$  and  $t = 4.60$  in the table

```
[t2(91:95),v(91:95)]
```

4.5000	0.5694
4.5500	0.0788
4.6000	-0.4109
4.6500	-0.8996
4.7000	-1.3873

indicates that  $t = 4.56$  at the bolt's apex. Now the commands

```
y = zeros(n+1,1);
h = 10/n;

for j = 2:n+1
    y(j) = y(j-1) + v(j-1)*h + 0.5*(-.04*v(j-1) - 9.8)*h^2;
end
ye = 7350*(1 - exp(-t/25)) - 245*t;
[t, y(1:n/10:n+1), ye]
```

generate the table:



$t$	approx. $y$	actual $y$
0	0	0
1	43.1974	43.1976
2	75.0945	75.0949
3	96.1342	96.1348
4	106.7424	106.7432
5	107.3281	107.3290
6	98.2842	98.2852
7	79.9883	79.9895
8	52.8032	52.8046
9	17.0775	17.0790
10	-26.8540	-26.8523

We see at least 2-decimal place agreement between approximate and actual values of  $y$ . Finally, interpolation between  $t = 9$  and  $t = 10$  here suggests that  $y = 0$  just after  $t = 9.4$ . Then interpolation between  $t = 9.40$  and  $t = 9.45$  in the table

`[t2(187:191), y(187:191)]`

9.3000	4.7448
9.3500	2.6182
9.4000	0.4713
9.4500	-1.6957
9.5000	-3.8829

indicates that the bolt is aloft for about 9.41 seconds.

30. Now our differential equation is described by the MATLAB function

```
function vp = vpbolt2(t,v)
vp = -0.0011*v.*abs(v) - 9.8;
```

Then the commands

```
n = 200;
[t1,v1] = ode('rk','vpbolt2',0,10,49,n);
n = 2*n;
[t2,v] = ode('rk','vpbolt2',0,10,49,n);
t = (0:10)';
ve = zeros(size(t));
ve(1:5) = 94.388*tan(0.478837 - 0.103827*t(1:5));
ve(6:11) = -94.388*tanh(0.103827*(t(6:11)-4.6119));
```

```
[t, v1(1:n/20:1+n/2), v(1:n/10:n+1), ve]
```

generate the table:

$t$	with $n = 200$	with $n = 400$	actual $v$
0	49.0000	49.0000	49.0000
1	37.1548	37.1548	37.1547
2	26.2430	26.2430	26.2429
3	15.9456	15.9456	15.9455
4	6.0046	6.0046	6.0045
5	-3.8015	-3.8015	-3.8013
6	-13.5101	-13.5101	-13.5100
7	-22.9354	-22.9354	-22.9353
8	-31.8985	-31.8985	-31.8984
9	-40.2559	-40.2559	-40.2559
10	-47.9071	-47.9071	-47.9071

We notice first that the final three columns almost agree to the 4 displayed decimal places. Scanning the last column for sign changes in  $v$ , we suspect that  $v = 0$  (at the bolt's apex) occurs just after  $t = 4.6$  sec. Then interpolation between  $t = 4.600$  and  $t = 4.625$  in the table

```
[t2(185:189),v(185:189)]
```

4.6000	0.1165
4.6250	-0.1285
4.6500	-0.3735
4.6750	-0.6185
4.7000	-0.8635

indicates that  $t = 4.61$  at the bolt's apex. Now the commands

```
y = zeros(n+1,1);
h = 10/n;
for j = 2:n+1
    y(j) = y(j-1) + v(j-1)*h + 0.5*(-.04*v(j-1) - 9.8)*h^2;
end
ye = zeros(size(t));
ye(1:5) = 108.465+909.091*log(cos(0.478837 - 0.103827*t(1:5)));
ye(6:11) = 108.465-909.091*log(cosh(0.103827*(t(6:11)-4.6119)));
[t, y(1:n/10:n+1), ye]
```

generate the table:

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$t$	Approx. $y$	Actual $y$
0	0	0.0001
1	42.9881	42.9841
2	74.6217	74.6197
3	95.6719	95.6742
4	106.6232	106.6292
5	107.7206	107.7272
6	99.0526	99.0560
7	80.8027	80.8018
8	53.3439	53.3398
9	17.2113	17.2072
10	-26.9369	-26.9363

We see almost 2-decimal place agreement between approximate and actual values of  $y$ . Finally, interpolation between  $t = 9$  and  $t = 10$  here suggests that  $y = 0$  just after  $t = 9.4$ . Then interpolation between  $t = 9.400$  and  $t = 9.425$  in the table

[t2 (377 : 381) , y (377 : 381) ]

9.4000	0.4740
9.4250	-0.6137
9.4500	-1.7062
9.4750	-2.8035
9.5000	-3.9055

indicates that the bolt is aloft for about 9.41 seconds.